Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

AN ELEMENTARY EXPOSITION OF THE THEORY OF THE GAMMA FUNCTION.

BY J. L. W. V. JENSEN.

Authorized translation from the Danish by T. H. Gronwall.[1]

Although there exists a very considerable literature concerning the Gamma function, it is but recently that a connected exposition of the properties of this function has appeared, with a fairly complete bibliography[2, 3]. However, the monograph referred to does not seem to lay any stress on a rigorous exposition, and the treatment is neither carried out from any definite point of view, nor based on the simplest foundations. The present paper may therefore not be entirely devoid of interest, since it gives a short, but rigorous and fairly complete theory of the Gamma and allied functions, based on the elementary theory of functions, i. e., that part of the theory of infinite series and products, and particularly power series, which may be treated in a simple and natural way without the aid of the calculus. Thus I shall deal below with the properties of certain functions of a complex variable, while the connection with definite integrals must fall beyond the scope of the present paper; the transition to these may however be made with the utmost facility at almost any stage, as I may perhaps show on another occasion.[4]

1. **Definitions.** When we attempt to determine a single-valued function $f(s)$ satisfying the functional equation

$$f(s+1) = sf(s), \tag{1}$$

it is necessary to inquire first if this problem has more than one solution.

---

[1] Gammafunktionens Theori i elementær Fremstilling. Nyt Tidsskrift for Mathematik, Afdeling B, vol. 2 (1891), pp. 33–35, 57–72, 83–84.

It is hoped that a translation of this monograph, which combines the merits of conciseness and elegance in an unusual degree, will be of general interest. The footnotes enclosed in square brackets [ ] have been added by the translator, and contain additional references to the literature, some propositions supplementary to the text, and finally explanations, in the infrequent cases where such have been thought advisable.

[2] Brunel, G. Monographie de la fonction Gamma. Mémoires de la société des sciences physiques et naturelles de Bordeaux, ser. 3, vol. 3 (1886), pp. 1–184.

[3] [Of the literature published since the appearance of the original, we shall only mention here Brunel's article on definite integrals in Encyclopedie der Math. Wissensch. and particularly Nielsen, N., Handbuch der Theorie der Gammafunktion, Leipzig, Teubner, 1906. The latter contains a very complete bibliography.]

[4] [This project was unfortunately never carried out.]

Let $F(s)$ denote, for the moment, some definite and single-valued solution, and write $f(s) = p(s) \cdot F(s)$; it is then seen at once that the relation

$$p(s+1) = p(s)$$

constitutes the necessary and sufficient condition that $f(s)$ shall satisfy (1). In other words, whenever we have found one solution of the functional equation, there exists an infinity, since any solution retains its character as such upon multiplication by an arbitrary periodic function having the additive period unity, and conversely it is evident that every possible solution is obtainable in this manner.

We[5] are led to the discovery of a particular solution of (1) by the following considerations.

For any particular solution $F(s)$ that may exist, we have, by (1),

$$\begin{aligned} F(s) &= \frac{F(s+1)}{s} = \frac{F(s+2)}{s(s+1)} = \cdots \\ &= \frac{F(s+n)}{s(s+1)\cdots(s+n-1)} \end{aligned} \tag{2}$$

where n, as throughout the following discussion, denotes a positive integer, and s a variable which never equals a negative integer or zero but may take any other value real or *complex.*

Moreover when s is a positive integer, we have

$$F(s+n) = (s+n-1)(s+n-2)\cdots n \cdot (n-1)!F(1),$$

and therefore

$$\lim_{n \to \infty} \frac{F(s+n)}{(n-1)!n^s} = F(1). \tag{a}$$

Therefore, since

$$\frac{F(s+n)}{s(s+1)\cdots(s+n-1)} = \frac{F(s+n)}{(n-1)!n^s} \cdot \frac{(n-1)!n^s}{s(s+1)\cdots(s+n-1)}, \tag{b}$$

we have, at least for positive integral values of s,

$$F(s) = F(1) \cdot \lim_{n \to \infty} \frac{(n-1)!n^s}{s(s+1)\cdots(s+n-1)}. \tag{c}$$

But, as we proceed to show, the second factor on the right has a definite value for every finite value of s, real or complex, except $0, -1, -2, \cdots$; it being understood that here and in what follows n^s is the number $e^{s \log n}$, where log n denotes the real value of the logarithm of n. For

---

[5] The discussion from this point to that similarly marked below replaces the corresponding discussion in Jensen's paper. The proof of the uniform convergence of the infinite product which defines $\Gamma(s)$, however, is that given by Jensen.

$$\frac{(n-1)!n^s}{s(s+1)\cdots(s+n-1)} = \frac{1}{s}\prod_{\nu=1}^{n-1}\frac{\left(1+\frac{1}{\nu}\right)^s}{1+\frac{s}{\nu}},$$

and the infinite product which is the limit, as n approaches ∞, of the right member of this equation, converges uniformly for every s bounded in absolute value and different from $0, -1, -2, \cdots$. An infinite product of the form $\prod_{\nu=1}^{\infty}(1+a_\nu)$ will converge if the series $\sum_{\nu=1}^{\infty} a_\nu$ is absolutely convergent and therefore, in particular, if $\nu^2|a_\nu|$ approaches a finite limit. But, given any number $N > 1$, we have for any s such that $|s| < N$ and any $\nu > N$

$$\left|\frac{\left(1+\frac{1}{\nu}\right)^s}{1+\frac{s}{\nu}} - 1\right| = \frac{\left|\frac{s(s-1)}{2!}\cdot\frac{1}{\nu^2}+\frac{s(s-1)(s-2)}{3!}\cdot\frac{1}{\nu^3}+\cdots\right|}{\left|1+\frac{s}{\nu}\right|}$$

$$< \left(\frac{N(N+1)}{2!}\cdot\frac{1}{\nu^2}+\cdots\right)\Big/\left(1-\frac{N}{\nu}\right),$$

and the product of this last expression (which is independent of s) by ν^2 approaches the limit $\frac{1}{2}N(N+1)$ as ν increases indefinitely. Therefore the infinite product under consideration *converges*, and that *uniformly*, for every s bounded in absolute value, and this remains true when s is zero or a negative integer provided the corresponding infinite factor is omitted from the product.

The function defined by this infinite product is denoted by $\Gamma(s)$, so that

$$\Gamma(s) = \frac{1}{s}\prod_{\nu=1}^{\infty}\frac{\left(1+\frac{1}{\nu}\right)^s}{1+\frac{s}{\nu}} \tag{3}$$

or

$$\Gamma(s) = \lim_{n\to\infty}\frac{(n-1)!n^s}{s(s+1)\cdots(s+n-1)}. \tag{3'}$$

It has the properties[6]

---

[6] Weierstrass, K., Ueber die Theorie der analytischen Facultäten. Journ. für Math., vol. 51 (1856), pp. 1–60. Reprinted in Abhandlungen aus der Functionenlehre (Berlin, 1886), pp. 181–262 [and *Mathematische Werke*, vol. 1 (Berlin, 1894), pp. 153–211].

$$(4a)\qquad \Gamma(s+1) = s\Gamma(s),$$

$$(4b)\qquad \lim_{n\to\infty} \frac{\Gamma(s+n)}{(n-1)!n^s} = 1,$$

from which it also follows that

$$\Gamma(1) = 1 \quad\text{and}\quad \Gamma(n) = (n-1)!$$

For, by (3′),

$$\frac{\Gamma(s+1)}{\Gamma(s)} = \lim_{n\to\infty}\left[\frac{(n-1)!n^{s+1}}{(s+1)(s+2)\cdots(s+n)} \div \frac{(n-1)!n^s}{s(s+1)\cdots(s+n-1)}\right] = s,$$

and therefore $\Gamma(s+1) = s\Gamma(s)$. But, this having been established, we have, as in (2) and (*b*),

$$(d)\qquad \Gamma(s) = \frac{\Gamma(s+n)}{(n-1)!n^s}\cdot\frac{(n-1)!n^s}{s(s+1)\cdots(s+n-1)},$$

and therefore, since the limit of the second factor on the right as n approaches ∞ is itself $\Gamma(s)$, we have, for *all* values of s,

$$\lim_{n\to\infty}\frac{\Gamma(s+n)}{(n-1)!n^s} = 1.$$

That $\Gamma(1) = 1$ is seen at once from (3). It also follows from (4*a*) and (4*b*), since when s is a positive integer we may, by (4*a*), replace $\Gamma(s+n)$ by $(s+n-1)!\Gamma(1)$ in (4*b*), and (4*b*) then becomes $\Gamma(1) = 1$.

Conversely the two conditions (4*a*), (4*b*) serve to define $\Gamma(s)$. For (*d*) follows from (4*a*) and from (*d*) in turn it follows, by (4*b*), that $\Gamma(s)$ is the function defined by (3′).

Thus $\Gamma(s)$ is itself a particular solution of the initial functional equation, $f(s+1) = sf(s)$, (1). Notice also that if in (*c*) we replace $F(s)$ by $\Gamma(s)$ and therefore $F(1)$ by 1, (*c*) reduces to (3′).

Multiplying[5] (3′) by

$$1 = \lim_{n\to\infty}(n+a)^s/n^s,$$

where a is independent of n but may depend on s, it is seen that

$$(3'')\qquad \Gamma(s) = \lim_{n\to\infty}\frac{(n-1)!(n+a)^s}{s(s+1)\cdots(s+n-1)},$$

which is slightly more general in form than (3′) and will be used later.

Before proceeding further, we shall transform (3). It is well known that $1/(\nu+1) < \log(1+1/\nu) < 1/\nu$, or

$$0 < \frac{1}{\nu} - \log\left(1+\frac{1}{\nu}\right) < \frac{1}{\nu} - \frac{1}{\nu+1},$$

whence it follows that $\Sigma(1/\nu - \log(1 + 1/\nu))$ is convergent. On the other hand,

$$\sum_1^n \left(\frac{1}{\nu} - \log\left(1 + \frac{1}{\nu}\right)\right) = 1 + \tfrac{1}{2} + \cdots + \frac{1}{n} - \log(n + 1),$$

and hence we may write

$$\begin{aligned} C &= \sum_1^\infty \left(\frac{1}{\nu} - \log\left(1 + \frac{1}{\nu}\right)\right) \\ &= \lim_{n\to\infty}\left[1 + \tfrac{1}{2} + \cdots + \frac{1}{n} - \log(n + 1)\right], \end{aligned}$$

which is known as Euler's constant.[7] Its value correct to ten decimals is $0.5772156649\cdots$.[8] Consequently

$$e^{-Cs} = e^{\sum_1^\infty \left(s\log\left(1+\frac{1}{\nu}\right) - \frac{s}{\nu}\right)} = \prod_1^\infty \left(1 + \frac{1}{\nu}\right)^s e^{-s/\nu},$$

and dividing this by (3), we obtain

$$\frac{1}{\Gamma(s)} = e^{Cs} \cdot s \prod_1^\infty \left(1 + \frac{s}{\nu}\right) e^{-s/\nu}. \tag{5}$$

As stated above, the condition (4*b*) was given by Weierstrass in his famous memoir on analytic factorials,[6] which may be considered as an unsurpassed model of rigorous and clear exposition. Equation (3) is due to Euler,[9] who unfortunately soon replaced this excellent definition by definite integrals[10]; in consequence, several of the formal properties of the Gamma function escaped his attention. (3′) is due to Gauss[11] who undoubtedly was not familiar with Euler's expression. (5) is due to [Schlömilch and] Newman[12], and was also established by Weierstrass.[6] Euler's constant, referred to above, is sometimes, without any justification, called Mascheroni's constant. What we have denoted here by $\Gamma(s)$ according to

---

[7] Euler, L. De progressionibus harmonicis observationes. Comment. Acad. Petrop. vol. 7 (1734–1735, published 1740), p. 156.

[8] [The arithmetical character of this constant is entirely unknown; it has not even been shown to be irrational.]

[9] Euler, L. Letter to Goldbach, Oct. 13, 1729. Correspondence math. et phys. de quelques célèbres géomètres du 18^e siècle, publiée par Fuss, vol. 1 (St. Pétersburg, 1843), p. 1.

[10] Euler, L. De progressionibus transcendentibus seu quarum termini generales algebraice dari nequeunt. Comment. Acad. Petrop., vol. 5 (1730–31, published 1738), pp. 36–57.

[11] Gauss, C. F. Disquisitiones generales circa seriem infinitam etc. Comment. Gotting., vol. 2 (1813), pp. 1–46. Reprinted in Werke, vol. 3 (1876), pp. 122–162. [German translation by H. Simon, Berlin, 1888.]

[12] [Schlömilch, O. Einiges über die Eulerschen Integrale der zweiten Art. Grunert Archiv, vol. 4 (1844), pp. 167–174.] Newman, On $\Gamma(a)$ especially when a is negative. Cambridge and Dublin math. Journal, vol. 3 (1848), pp. 57–60.

Legendre,[13] was written $[s-1]$ by Euler, $\Pi(s-1)$ by Gauss and $1/Fc(s)$ by Weierstrass.

2. Properties. From (5) it follows immediately, by well known elementary propositions, that $1/\Gamma(s)$ *may be expanded in a power series convergent for all finite values of* s, or that $1/\Gamma(s)$ is an entire transcendental function, with the simple zeros $0, -1, -2, \cdots$.[6] The function $\Gamma(s)$ is consequently a single-valued analytic function of s with the poles of order one $0, -1, -2, \cdots$, and it is seen at the same time that the equation $\Gamma(s) = 0$ has no roots. All this also follows from the fact that the product on the right side of (4) was shown to be uniformly convergent.

Since $\sin \pi s$ has the zeros $0, \pm 1, \pm 2, \cdots$, one is led to consider the function $1/(\Gamma(s)\Gamma(1-s))$ which has the same zeros. By reason of (4a) and (3) or (5) it is found that

$$\frac{1}{\Gamma(s)\Gamma(1-s)} = -\frac{1}{s\Gamma(s)\Gamma(-s)} = s\prod_{1}^{\infty}\left(1-\frac{s^2}{\nu^2}\right).$$

This product, however, equals $(1/\pi)\sin \pi s$, by a well known theorem due to Euler and proved in the elements of the theory of functions, and consequently

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}, \tag{6}$$

which is also due to Euler.[14] For $s = \frac{1}{2}$, this formula gives

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}, \tag{7}$$

where the square root is positive, since the product expansions (3) and (5) both show that *when s is real and positive, the same is true of* $\Gamma(s)$. Equation (7) was discovered by Euler as early as 1729.[9] Since it is known that[15]

---

[13] [Legendre, A. M. Recherches sur diverses sortes d'intégrales définies, Mém. de l'Institut, vol. 10 (1809), pp. 416–509.]

[14] [Euler, L. Evolutio formulæ integralis, etc. Novi Comment. Acad. Petrop. vol. 16 (1771, published 1772), pp. 91–139.]

[15] {For when h is odd,

$$\sin\frac{\pi}{h}\sin\frac{2\pi}{h}\cdots\sin\frac{(h-1)\pi}{h} = \sin^2\frac{\pi}{h}\sin^2\frac{2\pi}{h}\cdots\sin^2\frac{h-1}{2h}\pi$$
$$= 2^{(1-h)/2}\left(1-\cos\frac{2\pi}{h}\right)\left(1-\cos\frac{4\pi}{h}\right)\cdots\left(1-\cos\frac{(h-1)\pi}{h}\right). \tag{a}$$

But h being odd we have

$$x^h + 1 = (x+1)\prod_{i=1}^{(h-1)/2}\left(x^2 - 2x\cos\frac{2i-1}{h}\pi + 1\right)$$

and if both members of this identity be divided by $x+1$ and x be then set equal to 1 in the result we obtain

$$h = 2^{(h-1)/2}\left(1-\cos\frac{2\pi}{h}\right)\left(1-\cos\frac{4\pi}{h}\right)\cdots\left(1-\cos\frac{(h-1)\pi}{h}\right). \tag{b}$$

By substituting (b) in (a) we at once obtain the formula of the text. The proof for h even is similar.}

$$2^{1-h} \cdot h = \sin\frac{\pi}{h}\sin\frac{2\pi}{h}\cdots\sin\frac{(h-1)\pi}{h},$$

where h is a positive integer, we find by making $s = 1/h, 2/h, \cdots, (h-1)/h$ in (6), multiplying the resulting equations and extracting the square root,

$$\Gamma\left(\frac{1}{h}\right)\Gamma\left(\frac{2}{h}\right)\cdots\Gamma\left(\frac{h-1}{h}\right) = (2\pi)^{(h-1)/2}h^{-1/2}, \tag{8}$$

which is also due to Euler. In § 4, we shall give another proof of this equation without using the properties of the sine. Writing

$$(s, n) = \frac{\Gamma(n)n^s}{s(s+1)\cdots(s+n-1)},$$

it follows from (3′) that $\lim_{n\to\infty}(s, n) = \Gamma(s)$; it is readily seen that

$$\frac{(s, n)\left(s+\frac{1}{h}, n\right)\cdots\left(s+\frac{h-1}{h}, n\right)}{(hs, hn)} = \frac{\Gamma(n)^h n^{(h-1)/2}h^{hn}}{\Gamma(nh)}\cdot h^{-hs},$$

and since the left side of this equation has a definite limit for $n \to \infty$, the same must be true for the right side, so that

$$\frac{\Gamma(s)\Gamma\left(s+\frac{1}{h}\right)\cdots\Gamma\left(s+\frac{h-1}{h}\right)}{\Gamma(hs)} = k\cdot h^{-hs},$$

where the constant k may be determined with the aid of (8) by making $s = 1/h$, which gives $k = (2\pi)^{(h-1)/2}h^{1/2}$. Thus we have proved Gauss's[11] theorem[16, 17]

$$\Gamma(s)\Gamma\left(s+\frac{1}{h}\right)\cdots\Gamma\left(s+\frac{h-1}{h}\right) = h^{1/2-hs}(2\pi)^{(h-1)/2}\Gamma(hs). \tag{9}$$

---

[16] [The special case $h = 2$ is due to Legendre.[13]]

[17] [This may be generalized as follows: h and k being positive integers, (9) gives

$$\Gamma\left(hs+\frac{h\nu}{k}\right) = h^{h(s+\nu/k)-1/2}(2\pi)^{-(h-1)/2}\prod_{\mu=0}^{h-1}\Gamma\left(s+\frac{\nu}{k}+\frac{\mu}{h}\right),$$

whence

$$\prod_{\nu=0}^{k-1}\Gamma\left(hs+\frac{h\nu}{k}\right) = h^{hks+[h(k-1)]/2-k/2}(2\pi)^{-[k(h-1)]/2}\prod_{\nu=0}^{k-1}\prod_{\mu=0}^{h-1}\Gamma\left(s+\frac{\nu}{k}+\frac{\mu}{h}\right).$$

Dividing this equation by the one obtained by interchanging h and k, μ and ν, it is seen that

$$\frac{\prod_{\nu=0}^{k-1}\Gamma\left(hs+\frac{h\nu}{k}\right)}{\prod_{\mu=0}^{h-1}\Gamma\left(ks+\frac{k\mu}{h}\right)} = \left(\frac{h}{k}\right)^{hks+(hk-h-k)/2}\cdot(2\pi)^{(k-h)/2}, \tag{9′}$$

which reduces to (9) for $k = 1$. Equation (9′) was obtained from the definite integral expression for log $\Gamma(s)$ by Winckler, A., Neue Theoreme zur Lehre von den bestimmten Integralen, Sitzungsber. Akad. Wien, vol. 21 (1856), pp. 389–426. Compare Nielsen,[3] pp. 196–198 and 326. The remark that (9′) is a consequence of (9) appears to be new.]

3. Evaluation of some infinite products by means of the Gamma function. In the notation introduced above, we have for a positive integer $m > n$,

$$\frac{(s, m)}{(s, n)} = \frac{\left(\frac{m}{n}\right)^s}{\left(1 + \frac{s}{n}\right)\left(1 + \frac{s}{n+1}\right) \cdots \left(1 + \frac{s}{m-1}\right)},$$

whence, letting m and n increase indefinitely in a quite arbitrary fashion,

$$\frac{\Gamma(s)}{\Gamma(s)} = 1 = \lim \frac{\left(\frac{m}{n}\right)^s}{\left(1 + \frac{s}{n}\right)\left(1 + \frac{s}{n+1}\right) \cdots \left(1 + \frac{s}{m-1}\right)}.$$

Assuming in particular that m/n *tends toward a definite limit,* we have

$$\lim \left(\frac{m}{n}\right)^s = \lim \prod_n^{m-1} \left(1 + \frac{s}{\nu}\right). \tag{10}$$

By division of Gamma functions of different variables we obtain from either of equations (3) and (5)

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + s)\Gamma(\beta - s)} = \prod_0^\infty \left(1 + \frac{s}{\alpha + \nu}\right)\left(1 - \frac{s}{\beta + \nu}\right). \tag{11}$$

Writing, according to Mellin,[18] $R(s) = a_1 s + a_2 s^2 + \cdots + a_n s^n$, and denoting the n roots of the equation

$$\rho^n(1 + R(1/\rho)) = \rho^n + a_1\rho^{n-1} + \cdots + a_n = 0$$

by $\rho_1, \rho_2, \cdots, \rho_n$, it follows from (5) that

$$\frac{\Gamma(\alpha)^n}{\Gamma(\alpha - \rho_1 s)\Gamma(\alpha - \rho_2 s) \cdots \Gamma(\alpha - \rho_n s)} = e^{Ca_1 s}\left(1 + R\left(\frac{s}{\alpha}\right)\right)\prod_1^\infty \left(1 + R\left(\frac{s}{\alpha + \nu}\right)\right) e^{-a_1 s/\nu}. \tag{12}$$

We note the special case $R(s) = - s^n$, which gives for $\alpha = 1$

$$\frac{1}{\Gamma(1 - s)\Gamma(1 - e^{2\pi i/n}s)\Gamma(1 - e^{4\pi i/n}s) \cdots \Gamma(1 - e^{[(n-1)2\pi i]/n}s)} = \prod_1^\infty \left(1 - \left(\frac{s}{\nu}\right)^n\right),$$

a formula indicated by Liouville.[19]

---

[18] Mellin, H., Eine Verallgemeinerung der Gleichung $\Gamma(x)\Gamma(1 - x) = \pi/\sin \pi x$. Acta Math., vol. 3 (1883), pp. 102–104. Über gewisse durch die Gammafunktion ausdrückbare Producte, ibid., pp. 322–324.

[19] Liouville, J., Note sur la fonction gamma de Legendre. Comptes rendus vol. 35 (1852); pp. 317–322, and Journal de math., vol. 17 (1852), pp. 448–453.

When $R(s)$ is an *even* function of s, the convergent product

$$\prod_0^\infty\left(1+R\left(\frac{s}{\nu+1}\right)\right)$$

is expressible by trigonometric and exponential functions alone, on account of (6). This result is also easily derived from Euler's product expansion of $\sin \pi x$ without introducing the Gamma function.

4. Expansion of $\log \Gamma(s)$. Any finite and definite limit $\lim\limits_{n\to\infty} u_n$ may be rewritten as an infinite *convergent* series $u_h + \sum\limits_h^\infty (u_{\nu+1} - u_\nu)$. This remark will now be used to transform equation (3''), by taking logarithms on both sides and making $a = s$. It should be noted that in the following any logarithm is defined as that branch of the logarithmic function which is real for real and positive values of the variable. When s is represented in the complex plane in the familiar way, both sides of the equation referred to will then be single-valued as long as we do not make a circuit of any of the points $0, -1, -2, \cdots$. Thus, writing

$$u_n = \log \Gamma(n) + s \log(n+s) - \log s - \log(s+1) - \cdots - \log(s+n-1),$$

we obtain

$$\begin{aligned} u_{\nu+1} - u_\nu &= \log \nu + s \log \frac{s+\nu+1}{s+\nu} - \log(s+\nu) \\ &= \left[(s+\nu+\tfrac{1}{2}) \log \frac{s+\nu+1}{s+\nu} - 1\right] - \left[(\nu+\tfrac{1}{2}) \log \frac{\nu+1}{\nu} - 1\right] \\ &\quad - \left[(\nu+\tfrac{1}{2}) \log \frac{s+\nu+1}{\nu+1} - (\nu-\tfrac{1}{2}) \log \frac{s+\nu}{\nu}\right]. \end{aligned}$$

Now the expression inside the last of the three square brackets is the general term of a *convergent* series, since

$$\lim_{n\to\infty} (n+\tfrac{1}{2}) \log \frac{s+n+1}{n+1} = \lim_{n\to\infty} (n+\tfrac{1}{2}) \log\left(1+\frac{s}{n+1}\right) = s,$$

and the sum of this series for $\nu = 1, 2, 3, \cdots$, therefore, in consequence of the remark made above, equals $s - \frac{1}{2}\log(1+s)$. Furthermore,

$\Sigma\left[(\nu+\tfrac{1}{2}) \log \frac{\nu+1}{\nu} - 1\right]$ converges, since for $\nu > 1$,

$$(\nu+\tfrac{1}{2}) \log\left(1+\frac{1}{\nu}\right) - 1 = \frac{1}{12\nu^2} - \frac{1}{12\nu^3} + \cdots,$$

and the product of this expression by ν^2 is bounded. Consequently $\Sigma\left[(s+\nu+\frac{1}{2})\log\frac{s+\nu+1}{s+\nu}-1\right]$ must be convergent, since $\Sigma(u_{\nu+1}-u_\nu)$ converges, and we may write

$$\omega(s)=\sum_0^\infty\left[(s+\nu+\tfrac{1}{2})\log\frac{s+\nu+1}{s+\nu}-1\right]. \tag{13}$$

It is then seen at once that

$$\begin{aligned}\log\Gamma(s)&=u_1+\sum_1^\infty(u_{\nu+1}-u_\nu)\\&=[s\log(1+s)-\log s]+\left[\omega(s)-(s+\tfrac{1}{2})\log\frac{s+1}{s}+1\right]\\&\qquad\qquad-\omega(1)-[s-\tfrac{1}{2}\log(1+s)]\\&=(s-\tfrac{1}{2})\log s-s+1-\omega(1)+\omega(s).\end{aligned}$$

It remains to determine the constant $1-\omega(1)$, which we denote by k' for brevity, and to this purpose we observe that $\lim\limits_{n\to\infty}\omega(s+n)=0$, since $\omega(s+n)$ is *the remainder in the convergent series* $\omega(s)$, counted from the nth term. Making $s=n$ and $s=2n$ respectively, it is found that

$$\begin{aligned}2\log\Gamma(n)&=(2n-1)\log n-2n+2k'+2\omega(n),\\\log\ \Gamma(2n)&=(2n-\tfrac{1}{2})\log 2n-2n+k'+\omega(2n),\end{aligned}$$

whence by subtraction

$$\begin{aligned}k'+2\omega(n)-\omega(2n)&=\log\frac{\Gamma(n)^2n^{1/2}2^{2n-1/2}}{\Gamma(2n)}\\&=\log\frac{2^{1/2}\Gamma(n)n^{1/2}}{\frac{1}{2}(\frac{1}{2}+1)\cdots(\frac{1}{2}+n-1)},\end{aligned}$$

and increasing n indefinitely

$$k'=\log[2^{1/2}\Gamma(\tfrac{1}{2})]=\log\sqrt{2\pi}.$$

Hence the final result becomes

$$\log\Gamma(s)=(s-\tfrac{1}{2})\log s-s+\log\sqrt{2\pi}+\omega(s). \tag{14}$$

The infinite series $\omega(s)$ is due to Gudermann.[20] As special cases of (14) we note, for $s=1$ and $s=\frac{1}{2}$ respectively,

$$\omega(1)=1-\log\sqrt{2\pi}\qquad\text{and}\qquad\omega(\tfrac{1}{2})=\tfrac{1}{2}(1-\log 2).$$

---

[20] Gudermann, C., Additamentum ad functionis $\Gamma(a)$ theoriam. Journal für Math., vol. 29 (1845), pp. 209–212.

We have seen above that $\omega(s)$ converges toward zero when s increases by positive integral increments toward infinity. Therefore, by (14), $(s - \frac{1}{2}) \log s - s + \log \sqrt{2\pi}$ becomes an *approximate expression* for $\log \Gamma(s)$ when s increases in the manner indicated. When s is a positive integer, we have, in particular, the approximation formula known as Stirling's[21] formula. Before proceeding to generalize this result, we shall give some applications of (14). We have

$$\Gamma(s) = \lim_{n\to\infty} \frac{\Gamma(n) n^s}{s(s+1)\cdots(s+n-1)}$$

$$= \lim_{n\to\infty} \frac{n^{n-1/2} \cdot e^{-n} \cdot \sqrt{2\pi} \cdot n^s \cdot e^{\omega(n)}}{s(s+1)\cdots(s+n-1)},$$

and consequently

$$\Gamma(s) = \sqrt{2\pi} \lim_{n\to\infty} \frac{n^{n+s-1/2} e^{-n}}{s(s+1)\cdots(s+n-1)}, \tag{15}$$

which is due to [Enneper and] Gilbert.[22] Furthermore we may determine by means of (14), as shown by Cauchy,[23] the limit k of

$$\frac{\Gamma(n)^h n^{h-1/2} h^{hn}}{\Gamma(nh)} = h^{1/2} (2\pi)^{h-1/2} e^{h\omega(n) - \omega(nh)},$$

or $k = h^{1/2}(2\pi)^{h-1/2}$. Thus Gauss's theorem (9), and (8) as a particular case of it, is proved anew. The most elegant proof is however obtained from (15). The notation $\omega(s)$ is due to Catalan; Binet[24] denoted this function by $\mu(s)$ and Cauchy by $\varpi(s)$.

5. Investigation of $\omega(s)$ and $\log \Gamma(s)$ for large values of $|s|$. When $|\alpha + 1| > 1$, we have

$$(\alpha + \tfrac{1}{2}) \log\left(1 + \frac{1}{\alpha}\right) - 1 = -\left[(\alpha + 1) - \tfrac{1}{2}\right] \log\left(1 - \frac{1}{\alpha + 1}\right) - 1$$

$$= \sum_{\nu=2}^{\infty} \frac{\nu - 1}{2\nu(\nu + 1)} \frac{1}{(\alpha + 1)^\nu}.$$

---

[21] Stirling, J. Methodus differentialis sive tractatus de summatione et interpolatione serierum infinitarum, London, 1730, p. 135.

[22] [Enneper, A., Über die Function Π von Gauss mit complexem Argument. Diss. Göttingen 1856.] Gilbert, Ph., Recherches sur le développement de la fonction Γ et sur certaines intégrales définies qui en dépendent. Mém. Ac. Belgique, vol. 41 (1873), pp. 1–60.

[23] [Cauchy, A. L., Exercices de math., vol. 2 (Paris, 1827), pp. 91–92, and Nouveaux exercices, vol. 2 (Paris, 1841), pp. 407–408.]

[24] Binet, J., Mémoire sur les intégrales Eulériennes et sur leur application à la théorie des suites, ainsi qu'à l'évaluation des fonctions de grands nombres. Journal de l'École Polytechnique, cahier 27 (1839), pp. 123–343.

Now $\frac{\nu - 1}{2\nu(\nu + 1)} \leqq \frac{1}{12}$ for $\nu = 2, 3, 4, \cdots$, and consequently

$$\left|(\alpha + \tfrac{1}{2}) \log\left(1 + \frac{1}{\alpha}\right) - 1\right| < \frac{1}{12}\sum_{\nu=2}^{\infty} \frac{1}{|\,\alpha + 1\,|^{\nu}} = \frac{1}{12}\,\frac{1}{|\,\alpha + 1\,|(|\,\alpha + 1\,| - 1)}.$$

Applying this to the series (13), and observing that

$$\Sigma \frac{1}{|\,s + \nu + 1\,|\,(\,|\,s + \nu + 1\,| - 1)}$$

must be convergent since $\nu^2/[|\,s + \nu + 1\,|\,(|\,s + \nu + 1\,| - 1)]$ is bounded for ν increasing, it follows first that $\omega(s)$ may be expanded in the *convergent double series* (Binet[24])

$$(16)\qquad \begin{aligned} \omega(s) &= \sum_{\nu=2}^{\infty} \frac{\nu - 1}{2\nu(\nu + 1)} \sum_{\mu=1}^{\infty} \frac{1}{(s + \mu)^{\nu}} \\ &= \frac{1}{12}\sum_{\mu=1}^{\infty} \frac{1}{(s + \mu)^2} + \frac{1}{12}\sum_{\mu=1}^{\infty} \frac{1}{(s + \mu)^3} + \cdots, \end{aligned}$$

and second, that

$$|\,\omega(s)\,| < \frac{1}{12}\sum_{\nu=1}^{\infty} \frac{1}{|\,s + \nu\,|\,(\,|\,s + \nu\,| - 1)},$$

all this under the assumption that $|\,s + 1\,|$, $|\,s + 2\,|$, $\cdots$, are all greater than unity, or that s, when represented as usual in the complex plane, lies outside a sequence of circles with the radius 1 and the points -1, -2, -3, as centers, which we shall assume here for the sake of simplicity. (If s is inside or on the circumference of one or two of these circles, it is only necessary to omit one or two corresponding terms in the series for $\omega(s)$.)

Before proceeding to consider the last inequality for complex values of s, we shall make an observation connected with Stirling's formula. When s is real and positive, we have

$$0 < \omega(s) < \frac{1}{12}\sum_{1}^{\infty} \frac{1}{(s + \nu - 1)(s + \nu)} = \frac{1}{12}\sum_{1}^{\infty}\left(\frac{1}{s + \nu - 1} - \frac{1}{s + \nu}\right) = \frac{1}{12s},$$

and consequently by (14)

$$\log \Gamma(s) = (s - \tfrac{1}{2}) \log s - s + \log \sqrt{2\pi} + \frac{\theta}{12s}, \qquad 0 < \theta < 1,$$

a proposition which is frequently used and is commonly proved by the integral calculus.[25]

---

[25] [A more accurate formula is readily obtained by evaluating a lower bound for $\omega(s)$ in the same manner as the upper bound was found above. From (16) we obtain, s being real and positive,

$$\omega(s) > \frac{1}{12}\sum_{1}^{\infty} \frac{1}{(s + \nu)^2} > \frac{1}{12}\sum_{1}^{\infty} \frac{1}{(s + \nu)(s + \nu + 1)} = \frac{1}{12}\cdot\frac{1}{s + 1},$$

In the general case, let $(1 + \rho)$ be the shortest distance from $s = x + yi$ to any of the points $-1, -2, -3, \cdots$; then ρ is positive according to our hypothesis and $|s + \nu| > 1 + \rho$, so that

$$\frac{1}{|s+\nu|(|s+\nu|-1)} \leqq \frac{1}{|s+\nu|^2} \cdot \frac{1}{1 - \dfrac{1}{1+\rho}} = \frac{1+\rho}{\rho} \cdot \frac{1}{|s+\nu|^2}$$

$$= \frac{1+\rho}{\rho} \frac{1}{x^2 + y^2 + 2\nu x + \nu^2}.$$

First supposing *x positive*, and noting that $A^2 + B^2 \geqq \frac{1}{2}(A + B)^2$ when A and B are real and positive, the last expression will be less than

$$\frac{1+\rho}{\rho} \frac{1}{|s|^2 + \nu^2} \leqq \frac{1+\rho}{\rho} \frac{2}{(|s| + \nu)^2} < \frac{1+\rho}{\rho} \frac{2}{(|s| + \nu - 1)(|s| + \nu)}$$

$$= 2\frac{1+\rho}{\rho}\left(\frac{1}{|s| + \nu - 1} - \frac{1}{|s| + \nu}\right),$$

and consequently

$$\text{(I)} \qquad |\omega(s)| < \frac{1+\rho}{6\rho} \sum_{\nu=1}^{\infty} \left(\frac{1}{|s| + \nu - 1} - \frac{1}{|s| + \nu}\right)$$

$$= \frac{1+\rho}{6\rho} \cdot \frac{1}{|s|}, \qquad (x > 0).$$

On the other hand, let $-(m + 1) < x \leqq -m$, where m is a positive integer or zero; then

$$\sum_{\nu=1}^{\infty} \frac{1}{|s+\nu|^2} = \sum_{\nu=1}^{m-1} \frac{1}{|s+\nu|^2} + \sum_{\nu=m}^{m+1} \frac{1}{|s+\nu|^2} + \sum_{\nu=m+2}^{\infty} \frac{1}{|s+\nu|^2},$$

where, for $m = 0$ or 1, the first sum on the right side should be omitted. In the second sum on the right side we have

---

and consequently

$$\log \Gamma(s) = \left(s - \frac{1}{2}\right) \log s - s + \log \sqrt{2\pi} + \frac{1}{12} \cdot \frac{1}{s + \theta}, \qquad 0 < \theta < 1.$$

Retaining two terms in (16), we may even show that $0 < \theta < 1/2$ in the last formula, since

$$\omega(s) > \frac{1}{12} \sum_{1}^{\infty} \frac{1}{(s+\nu)^2} + \frac{1}{12} \sum_{1}^{\infty} \frac{1}{(s+\nu)^3}$$

and, for x real and > 1,

$$\frac{1}{x^2} + \frac{1}{x^3} - \frac{1}{x - \frac{1}{2}} + \frac{1}{x + \frac{1}{2}} = \frac{4x^2 - x - 1}{x^3(4x^2 - 1)} > 0,$$

whence, making $x = s + \nu$,

$$\omega(s) > \frac{1}{12} \sum_{1}^{\infty} \left(\frac{1}{s + \nu - \frac{1}{2}} - \frac{1}{s + \nu + \frac{1}{2}}\right) = \frac{1}{12} \cdot \frac{1}{s + \frac{1}{2}}.$$

This proof is believed to be new; for a still closer limitation of θ, obtained by definite integrals, see Sonin, M. N., Sur les termes complémentaires de la formule sommatoire d'Euler et de celle de Stirling, Comptes rendus, vol. 108 (1889), pp. 725–727, and Annales de l'Ecole Normale, ser. 3, vol. 6 (1889), pp. 257–262.]

$$\sum_{\nu=m}^{m+1} \frac{1}{|s+\nu|^2} = \frac{1}{(x+m)^2+y^2} + \frac{1}{(x+m+1)^2+y^2} < \frac{1}{y^2} + \frac{1}{y^2};$$

making $\nu = m - \mu$ in the first sum, so that $\mu = m-1, m-2, \cdots, 2, 1$, and $\nu = m + 1 + \mu$ in the third sum, so that $\mu = 1, 2, 3, \cdots$, we find, since $x + m \leqq 0$, $x + m + 1 > 0$,

$$\sum_{\nu=1}^{m-1} \frac{1}{|s+\nu|^2} = \sum_{\mu=1}^{m-1} \frac{1}{(\mu-(x+m))^2+y^2} \leqq \sum_{\mu=1}^{m-1} \frac{1}{\mu^2+y^2}$$

$$< \sum_{\mu=1}^{\infty} \frac{1}{\mu^2+y^2} \leqq \sum_{\mu=1}^{\infty} \frac{2}{(\mu+|y|)^2}$$

$$< 2\sum_{\mu=1}^{\infty} \frac{1}{(|y|+\mu-1)(|y|+\mu)} = \frac{2}{|y|};$$

$$\sum_{\nu=m+2}^{\infty} \frac{1}{|s+\nu|^2} = \sum_{\mu=1}^{\infty} \frac{1}{(\mu+(x+m+1))^2+y^2}$$

$$< \sum_{\mu=1}^{\infty} \frac{1}{\mu^2+y^2} < \frac{2}{|y|},$$

so that finally, adding the three sums,

$$\text{(II)} \qquad |\omega(s)| < \frac{1+\rho}{6\rho}\left(\frac{2}{|y|} + \frac{1}{y^2}\right), \qquad (x \leqq 0).^{26}$$

Since $(1+\rho)/\rho$ is bounded, it is now seen immediately from (I) and (II) that $\omega(s)$ *converges uniformly toward zero, when the distance between* s *and the nearest point on the negative real axis increases indefinitely.* In particular, this condition is satisfied when $|s|$ increases indefinitely while the arc of s[27] is constant and different from π. When $|s|$ increases in the manner indicated, it therefore follows from (14) that

$$\text{(17)} \qquad \lim_{|s|\to\infty} \frac{\Gamma(s)}{s^{s-1/2}e^{-s}\sqrt{2\pi}} = 1,$$

where that branch of the multiple-valued function $s^{s-1/2}$ is taken which is real for real and positive values of s. This equation may be used to extend the domain of validity of (4*b*); it is readily seen that, when t is any complex quantity,

$$\lim_{|s|\to\infty} \frac{\Gamma(s+t)}{\Gamma(s)s^t} = 1.$$

We have used above the expansion of $(\alpha + \frac{1}{2}) \log (1 + 1/\alpha) - 1$ in

---

[26] [The form and proof of (II) have been slightly changed from the original.]

[27] [When s is written in the form $s = |s| e^{\theta i}$, θ is the arc of s].

powers of $1/(\alpha + 1)$; the expansion in powers of $1/\alpha$ would have served the same purpose. In fact, for $|\alpha| > 1$,

$$(\alpha + \tfrac{1}{2}) \log \left(1 + \frac{1}{\alpha}\right) - 1 = \sum_{\nu=2}^{\infty} \frac{(-1)^\nu(\nu - 1)}{2\nu(\nu + 1)} \frac{1}{\alpha^\nu},$$

whence we deduce, instead of (16), the expansion

$$\begin{aligned} \omega(s) &= \sum_{\nu=2}^{\infty} \frac{(-1)^\nu(\nu - 1)}{2\nu(\nu + 1)} \sum_{\mu=0}^{\infty} \frac{1}{(s + \mu)^\nu} \\ &= \frac{1}{12} \sum_{\mu=0}^{\infty} \frac{1}{(s + \mu)^2} - \frac{1}{12} \sum_{\mu=0}^{\infty} \frac{1}{(s + \mu)^3} + \cdots, \end{aligned} \tag{18}$$

valid for $|s|$, $|s + 1|$, $|s + 2|$, $\cdots$ all greater than unity. This formula is also due to Binet.[28] Neither (16) nor (18) is adapted to the numerical calculation of $\omega(s)$ (and $\log \Gamma(s)$ through (14)); expansions suitable for this purpose will be derived in § 14.

6. The function $\psi(s)$. Definition and properties. It follows from (3) that

$$\log \frac{\Gamma(s + t)}{\Gamma(s)} = -\log \left(1 + \frac{t}{s}\right) + \sum_{1}^{\infty} \left[t \log \left(1 + \frac{1}{\nu}\right) - \log \left(1 + \frac{t}{s + \nu}\right)\right],$$

which series is uniformly convergent, since it is derived from uniformly convergent infinite products. When s is not equal to zero or a negative integer, $\Gamma(s + t)$ may be expanded in a Taylor series $\Gamma(s + t) = \Gamma(s) + t\Gamma'(s) + \cdots$ for sufficiently small values of $|t|$. Substituting this on the left and expanding both sides in powers of t, we find by comparison of the coefficients of t

$$\frac{\Gamma'(s)}{\Gamma(s)} = -\frac{1}{s} + \sum_{1}^{\infty} \left[\log \left(1 + \frac{1}{\nu}\right) - \frac{1}{s + \nu}\right].$$

This function, which will be denoted by $\psi(s)$, may be expressed in the following forms, using the series

$$C = \sum_{1}^{\infty} \left[\frac{1}{\nu} - \log \left(1 + \frac{1}{\nu}\right)\right]$$

and

$$-\frac{1}{s} = \sum_{0}^{\infty} \left(\frac{1}{s + \nu + 1} - \frac{1}{s + \nu}\right),$$

$$\begin{aligned} \psi(s) &= -C - \frac{1}{s} + \sum_{1}^{\infty} \left(\frac{1}{\nu} - \frac{1}{s + \nu}\right) \\ &= -C + \sum_{0}^{\infty} \left(\frac{1}{\nu + 1} - \frac{1}{s + \nu}\right). \end{aligned} \tag{19}$$

---

[28] Binet, J., Abstract of paper quoted in [24], Comptes rendus, vol. 9 (1839), pp. 39–45.

These series are uniformly convergent for all finite values of s; for choosing an N so large that $N > |s|$, it is seen that for $\nu > N$

$$\left|\frac{1}{\nu+1} - \frac{1}{s+\nu}\right| = \left|\frac{s-1}{(\nu+1)(s+\nu)}\right| < \frac{N+1}{(\nu+1)(\nu-N)},$$

the last expression being independent of s and being the general term of a convergent series. Since

$$-\frac{1}{s} + \sum_{1}^{n-1}\left[\log\left(1+\frac{1}{\nu}\right) - \frac{1}{s+\nu}\right] = \log n - \frac{1}{s} - \frac{1}{s+1} - \cdots - \frac{1}{s+n-1},$$

we have

$$(19') \qquad \psi(s) = \lim_{n\to\infty}\left(\log n - \frac{1}{s} - \frac{1}{s+1} - \cdots - \frac{1}{s+n-1}\right),$$

and using $0 = \lim_{n\to\infty}[\log(n+a) - \log n]$, this may also be written

$$(19'') \quad \psi(s) = \lim_{n\to\infty}\left(\log(n+a) - \frac{1}{s} - \frac{1}{s+1} - \cdots - \frac{1}{s+n-1}\right).$$

From (19) we furthermore obtain

$$(19''') \qquad \psi(s) - \psi(t) = \sum_{0}^{\infty}\left(\frac{1}{t+\nu} - \frac{1}{s+\nu}\right) = \sum_{0}^{\infty}\frac{s-t}{(t+\nu)(s+\nu)},$$

which is somewhat more general in form than (19).

From either of equations (19) it is seen at once that

$$\psi(1) = -C,$$

and that

$$(20) \qquad \begin{aligned} \psi(s+1) - \psi(s) &= \frac{1}{s}, \\ \psi(s+n) - \psi(s) &= \frac{1}{s} + \frac{1}{s+1} + \cdots + \frac{1}{s+n-1}. \end{aligned}$$

Substituting the last expression in (19′), we find

$$(21) \qquad \lim_{n\to\infty}[\psi(s+n) - \log n] = 0,$$

which will be generalized below. Conversely, it is obvious that (19′) may be derived from (20) and (21), so that *these two equations are sufficient to define* $\psi(s)$.

The expansions of $\psi(s)$ being uniformly convergent, $\psi(s)$ is *a single-valued analytic function of* s *which has poles of the first order at the points* $s = 0, -1, -2, \cdots$, *but is holomorphic for all other finite values of* s. By (19‴) we have, for $t = 1 - s$,

$$\psi(s) - \psi(1-s) = -\sum_0^\infty\left(\frac{1}{s+\nu} - \frac{1}{\nu+1-s}\right)$$

$$= -\frac{1}{s} + \frac{1}{1-s} - \frac{1}{1+s} + \frac{1}{2-s} - \frac{1}{2+s} + \cdots,$$

and consequently, by a well-known theorem due to Euler,*

$$(22) \qquad \psi(s) - \psi(1-s) = -\pi\cot\pi s,$$

which could also be derived from (6) in the same way as (19) was deduced from (3). Writing, for brevity,

$$[s, n] = \log n - \frac{1}{s} - \frac{1}{s+1} - \cdots - \frac{1}{s+n-1},$$

it is readily seen that, if h be any positive integer,

$$h[hs, hn] - [s, n] - \left[s + \frac{1}{h}, n\right] - \cdots - \left[s + \frac{h-1}{h}, n\right] = h\log h,$$

whence, letting n increase indefinitely,

$$(23) \quad h\psi(hs) - \psi(s) - \psi\left(s + \frac{1}{h}\right) - \cdots - \psi\left(s + \frac{h-1}{h}\right) = h\log h.$$

These propositions could all be obtained by logarithmic differentiation of the corresponding ones for $\Gamma(s)$ (Gauss[6]). I have preferred the above elementary deduction, which is as simple as could be desired. What we have denoted here by $\psi(s)$, according to Cauchy, was written $\Psi(s-1)$ by Gauss,[11] $\varphi(s)$ and $z'(s)$ by Legendre[13] and $\lambda'(s)$ by Binet.[24]

The series for $\psi(s)$ being uniformly convergent, $\psi(s+t)$ may be expanded in ascending powers of t for $|t| < |s|, |s+1|, |s+2|, \cdots$, and this may be effected by expanding the individual terms on the right side of (19). By Taylor's formula, the comparison of the coefficients of t^m shows that

$$(24) \qquad \psi^{(m)}(s) = (-1)^{m+1}m!\sum_{\nu=0}^\infty \frac{1}{(s+\nu)^{m+1}}, \qquad (m \geqq 1).$$

These functions are readily seen to be completely defined by the equations

$$\psi^{(m)}(s+1) - \psi^{(m)}(s) = \frac{(-1)^m m!}{s^{m+1}}, \quad \text{and} \quad \lim_{n\to\infty}\psi^{(m)}(s+n) = 0.$$

---

* {For

$$\sin\pi s = \pi s\prod_{\nu=1}^\infty\left(1 - \frac{s^2}{\nu^2}\right).$$

Hence

$$-\pi\cot\pi s = -\frac{d}{ds}\log\sin\pi s = -\frac{1}{s} + \sum_{\nu=1}^\infty\left(\frac{1}{\nu-s} - \frac{1}{\nu+s}\right).\}$$

They furthermore satisfy relations analogous to (22) and (23), which we shall not, however, stop to prove.

7. Evaluation of some infinite series and products by means of $\psi(s)$. By the use of the functions $\psi(s)$, $\psi'(s)$, $\cdots$ which we have introduced, it is easy to find the sum of series of very general form. For instance, when $F(s)$ is a rational function of the form

$$F(s) = \frac{as^{n-1} + bs^{n-2} + \cdots + l}{s^n + \alpha s^{n-1} + \beta s^{n-2} + \cdots + \lambda},$$

the sum of the convergent series $\Sigma[F(s + \nu) - F(t + \nu)]$ is found by resolving $F(s + \nu)$ and $F(t + \nu)$ into a sum of partial fractions in the familiar way, thus decomposing the series into several others which may be evaluated individually by means of (19‴) and (24). When $a = 0$, the series $\Sigma F(s + \nu)$ becomes convergent. In the particular case when $F(s)$ is a rational function of s^2, it is seen incidentally that $\Sigma F(s + \nu)$ may be evaluated without the aid of $\psi(s)$, $\psi'(s)$, $\cdots$ by the exclusive use of Euler's partial fraction expansions for the trigonometric functions. As an example of the application of these remarks we may mention

$$\sum_0^\infty \frac{1}{(s + \nu - \rho_1)(s + \nu - \rho_2) \cdots (s + \nu - \rho_n)}$$
$$= - \frac{\psi(s - \rho_1)}{(\rho_1 - \rho_2)(\rho_1 - \rho_3) \cdots (\rho_1 - \rho_n)} - \frac{\psi(s - \rho_2)}{(\rho_2 - \rho_1)(\rho_2 - \rho_3) \cdots (\rho_2 - \rho_n)} - \cdots - \frac{\psi(s - \rho_n)}{(\rho_n - \rho_1)(\rho_n - \rho_2) \cdots (\rho_n - \rho_{n-1})},$$

where all of $\rho_1, \rho_2, \cdots, \rho_n$ are *distinct.*[*]

---

[*] {Let

$$f(s) = \prod_{i=1}^n (s - \rho_i).$$

By a theorem of Euler,

$$\sum_{i=1}^n 1/f'(\rho_i) = 0.$$

Hence

$$- \sum_{\nu=0}^\infty \frac{1}{(s + \nu - \rho_1) \cdots (s + \nu - \rho_n)} = - \sum_{\nu=0}^\infty \sum_{i=0}^n \frac{1}{f'(\rho_i)(s + \nu - \rho_i)}$$
$$= \sum_{\nu=0}^\infty \sum_{i=1}^n \left[\frac{1}{\nu + 1} - \frac{1}{s + \nu - \rho_i}\right] \frac{1}{f'(\rho_i)} = \sum_{i=1}^n \sum_{\nu=0}^\infty \left[\frac{1}{\nu + 1} - \frac{1}{s + \nu - \rho_i}\right] \frac{1}{f'(\rho_i)}$$
$$= \sum_{i=1}^n [\psi(s - \rho_i) + C] \frac{1}{f'(\rho_i)} = \sum_{i=1}^n \frac{\psi(s - \rho_i)}{f'(\rho_i)}.\}$$

It is seen from (5) that

$$\frac{\Gamma(s)}{\Gamma(s+t)} = e^{Ct}\frac{s+t}{s}\prod_{1}^{\infty}\frac{1+\dfrac{s+t}{\nu}}{1+\dfrac{s}{\nu}}e^{-t/\nu} = e^{Ct}\left(1+\frac{t}{s}\right)\prod_{1}^{\infty}\left(1+\frac{t}{s+\nu}\right)e^{-t/\nu}.$$

Furthermore, it follows from (19) that

$$e^{t\psi(s)} = e^{-Ct-t/s}\prod_{1}^{\infty}e^{t/\nu-[t/(s+\nu)]},$$

and consequently (Mellin[29])

$$e^{t\psi(s)}\frac{\Gamma(s)}{\Gamma(s+t)} = \prod_{0}^{\infty}\left(1+\frac{t}{s+\nu}\right)e^{-t/(s+\nu)}, \tag{25}$$

a generalization of (5) and reducing to the latter for $s = 1$. Making $t = 1$, we obtain

$$e^{\psi(s)} = s\prod_{0}^{\infty}\left(1+\frac{1}{s+\nu}\right)e^{-1/(s+\nu)}. \tag{26}$$

8. Investigation of $\psi(s)$ for large values of $|s|$. Writing

$$u_n = \log(n+s) - \frac{1}{s} - \frac{1}{s+1} - \cdots - \frac{1}{s+n-1}, \qquad u_0 = \log s,$$

it follows from (19″) that for $a = s$

$$\psi(s) = \lim_{n\to\infty} u_n = u_0 + \sum_{0}^{\infty}(u_{\nu+1} - u_\nu),$$

and since

$$u_{\nu+1} - u_\nu = \log\left(1+\frac{1}{s+\nu}\right) - \frac{1}{s+\nu},$$

it is seen that, making

$$\omega^*(s) = \sum_{0}^{\infty}\left(\frac{1}{s+\nu} - \log\left(1+\frac{1}{s+\nu}\right)\right), \tag{27}$$

we have

$$\psi(s) = \log s - \omega^*(s). \tag{28}$$

When $|\alpha+1| > 1$, then

$$\frac{1}{\alpha} - \log\left(1+\frac{1}{\alpha}\right) = \frac{1}{(\alpha+1)-1} + \log\left(1-\frac{1}{\alpha+1}\right) = \sum_{2}^{\infty}\left(1-\frac{1}{\nu}\right)\frac{1}{(\alpha+1)^{\nu}},$$

and consequently

$$\left|\frac{1}{\alpha} - \log\left(1+\frac{1}{\alpha}\right)\right| < \sum_{2}^{\infty}\frac{1}{|\alpha+1|^{\nu}} = \frac{1}{|\alpha+1|(|\alpha+1|-1)},$$

---

[29] Mellin, H., Om gammafunktionen. Öfversigt Akad. Stockholm, 1883, no. 5, pp. 3–20.

whence we conclude that for $| s + 1 |$, $| s + 2 |$, $\cdots$ all greater than unity,

$$| \omega^*(s) | < \sum_1^\infty \frac{1}{| s + \nu | (| s + \nu | - 1)},$$

a series which we have already studied in § 5. When $| s |$ increases in the manner indicated there, $\omega^*(s) \to 0$, or

$$\lim_{|s| \to \infty} [\psi(s) - \log s] = 0, \tag{29}$$

which extends the domain of validity of (21). For $| s + 1 |$, $| s + 2 |$, $\cdots$ all greater than unity we find

$$\omega^*(s) = \sum_{\nu=2}^\infty \left(1 - \frac{1}{\nu}\right) \sum_{\mu=1}^\infty \frac{1}{(s + \mu)^\nu}. \tag{30}$$

On the other hand, the expansion

$$\frac{1}{\alpha} - \log\left(1 + \frac{1}{\alpha}\right) = \sum_2^\infty \frac{(-1)^\nu}{\nu} \frac{1}{\alpha^\nu}, \qquad (| \alpha | > 1),$$

gives

$$\omega^*(s) = \sum_{\nu=2}^\infty \frac{(-1)^\nu}{\nu} \sum_{\mu=0}^\infty \frac{1}{(s + \mu)^\nu}, \tag{31}$$

which is valid for $| s |$, $| s + 1 |$, $\cdots$ all greater than unity. Expansions more convenient than (30) and (31) for computing $\omega^*(s)$ numerically, will be derived in § 14.

Equation (29) may be used to determine approximately the roots of the transcendental equation $\psi(s) = 0$. According to (19), the imaginary part of $\psi(x + yi)$ equals

$$yi \cdot \sum_0^\infty \frac{1}{(x + \nu)^2 + y^2},$$

which cannot vanish unless $y = 0$, so that all the roots must be real. Furthermore it follows from (19‴) that

$$\psi(x) - \psi(x') = (x - x') \sum_0^\infty \frac{1}{(x + \nu)(x' + \nu)},$$

and every term in the series is positive when x and x' are both in one of the intervals $(0 \cdots + \infty)$, $(-1 \cdots 0)$, $(-2 \cdots -1)$, $\cdots$; since $\psi(+\infty) = +\infty$, $\psi(-n \pm 0) = \mp \infty$, it is seen that $\psi(x)$ increases steadily from $-\infty$ to $+\infty$ in any one of the intervals in question, so that each interval contains one root only. The positive root lies between 1 and 2, since $\psi(1) = -C < 0$, $\psi(2) = 1 - C > 0$, and its value $1.46163 \cdots$ was computed by Legendre. Denoting the negative root in the interval

$(-n \cdots -n+1)$ by $-n + x_n$, $(0 < x_n < 1)$, it follows from (22) that

$$\pi \cot \pi x_n = \psi(n + 1 - x_n) = \log n + \delta_n,$$

where, according to (29), $\delta_n \rightarrow 0$ for $n \rightarrow \infty$. Therefore

$$x_n = \frac{1}{\pi} \arctan \frac{\pi}{\log n + \delta_n},$$

whence we conclude that $x_n \log n \rightarrow 1$. For large values of n, the root in the interval $(-n \cdots -n+1)$ is thus approximately equal to

$$-n + \frac{1}{\log n},$$

a result due to Hermite.[30] Remembering that $\Gamma(x)$ never equals zero, we see at once that $\Gamma'(x) = \Gamma(x)\psi(x) = 0$ must have the same roots as $\psi(x) = 0$, which is of importance when inquiring how $\Gamma(x)$ varies for real values of x.

9. Evaluation of $\psi(x)$ for rational values of x. On account of (20), it is sufficient to evaluate

$$\psi\left(\frac{p}{q}\right) = -C + \sum_{\nu=0}^{\infty}\left(\frac{1}{\nu+1} - \frac{q}{p + \nu q}\right),$$

when p/q is a *proper* fraction (p and q integers, $q > p > 0$). Instead of this expansion we shall consider the power series

$$S(t) = \sum_{\nu=0}^{\infty}\left(\frac{1}{\nu+1} - \frac{q}{p + \nu q}\right) t^{p+\nu q},$$

which is convergent for $t = 1$. By Abel's theorem on power series,[31]

---

[30] Hermite, Ch., Sur l'intégrale Eulérienne de seconde espèce. Journal für Math., vol. 90 (1881), pp. 332–338.

[31] ["When $\sum_0^\infty a_\nu$ is convergent, the power series $F(t) = \sum_0^\infty a_\nu t^\nu$ approaches the limit $\sum_0^\infty a_\nu$ when t increases through real positive values toward unity." The proof commonly given is this: $\sum_0^\infty a_\nu$ being convergent, the remainder after n terms $r_n = \sum_n^\infty a_\nu \rightarrow 0$ for $n \rightarrow \infty$, and hence there exists a positive quantity A such that $| r_n | < A$ for $n = 0, 1, 2, \cdots$ and to any $\epsilon > 0$ but as small as we please, there corresponds an integer N such that $| r_n | < \epsilon$ for $n > N$. Obviously

$$a_\nu = r_\nu - r_{\nu+1},$$

and consequently for $0 < t < 1$

$$\begin{aligned} F(t) - \sum_0^\infty a_\nu = \sum_1^\infty a_\nu(t^\nu - 1) &= \sum_1^\infty (r^\nu - r_{\nu+1})(t^\nu - 1) \\ &= \lim_{n\rightarrow\infty} [(r_1 - r_2)(t-1) + (r_2 - r_3)(t^2 - 1) + \cdots + (r_n - r_{n+1})(t^n - 1)] \\ &= \lim_{n\rightarrow\infty} [-r_1(1-t) - r_2(t - t^2) - \cdots - r_n(t^{n-1} - t^n)] + \lim_{n\rightarrow\infty} r_{n+1}(1 - t^n), \end{aligned}$$

$$\psi\left(\frac{p}{q}\right) = -C + \lim_{t\to 1} S(t).$$

We have, however, for $|t| < 1$,

$$\sum_0^\infty \frac{t^{p+\nu q}}{\nu+1} = -t^{p-q}\log(1-t^q),$$

and the problem thus reduces to the determination, for $|t| < 1$, of $q\sum_0^\infty \frac{t^{p+\nu q}}{p+\nu q}$, which consists of q times every qth term in the series

$$\sum_1^\infty \frac{t^\nu}{\nu} = -\log(1-t).$$

But given the power series $F(t) = \sum_0^\infty c_\nu t^\nu$, it is well known that, for $\theta = e^{2\pi i/q}$,

$$\sum_{\mu=0}^{q-1}\theta^{-\mu p}F(t\theta^\mu) = q(c_p t^p + c_{p+q}t^{p+q} + c_{p+2q}t^{p+2q} + \cdots),$$

since

$$\sum_{\mu=0}^{q-1}\theta^{\mu(\nu-p)} = \frac{\theta^{q(\nu-p)}-1}{\theta^{\nu-p}-1} = \frac{1-1}{\theta^{\nu-p}-1} = 0$$

when $\nu - p$ is not divisible by q, but

$$= \sum_{\mu=0}^{q-1} 1 = q$$

when $\nu - p$ is divisible by q. Hence we find, for $|t| < 1$,

$$S(t) = -t^{p-q}\log(1-t^q) + \sum_{\mu=0}^{q-1}\theta^{-\mu p}\log(1-t\theta^\mu)$$

---

and since $r_{n+1}\to 0$, we have

$$F(t) - \sum_0^\infty a_\nu = -\sum_1^\infty r_\nu(t^{\nu-1}-t^\nu) = -(1-t)\sum_1^N r_\nu t^{\nu-1} - \sum_{N+1}^\infty r_\nu(t^{\nu-1}-t^\nu).$$

For $0 < t < 1$, it is clear that $1 - t > 0$, $t^{\nu-1} - t^\nu > 0$, whence

$$\begin{aligned}\left|F(t) - \sum_0^\infty a_\nu\right| &\leqq (1-t)\sum_1^N |r_\nu|\, t^{\nu-1} + \sum_{N+1}^\infty |r_\nu|\,(t^{\nu-1}-t^\nu)\\ &< (1-t)\sum_1^N A\cdot 1 + \sum_{N+1}^\infty \epsilon(t^{\nu-1}-t^\nu)\\ &= (1-t)NA + \epsilon t^N < NA(1-t) + \epsilon\\ &< 2\epsilon \text{ for } 0 < 1-t < \epsilon/NA,\end{aligned}$$

so that $\left|F(t) - \sum_0^\infty a_\nu\right|$ can be made as small as we please by taking t positive and less than, but sufficiently close to, unity.]

$$= -t^{p-q}\log\frac{1-t^q}{1-t}+\sum_{\mu=1}^{q-1}\theta^{-\mu p}\log(1-t\theta^\mu)-(t^{p-q}-1)\log(1-t),$$

and letting t increase to unity along the real axis

$$\psi\left(\frac{p}{q}\right) = -C-\log q+\sum_{\mu=1}^{q-1}\theta^{-\mu p}\log(1-\theta^\mu).$$

Thus our problem is solved, and it only remains to transform the expression obtained. Replacing p by $q-p$, a new equation results, and adding it to the above, we find

$$\psi\left(\frac{p}{q}\right)+\psi\left(\frac{q-p}{q}\right) = -2C-2\log q+2\sum_{\mu=1}^{q-1}\cos\frac{2\pi\mu p}{q}\log(1-\theta^\mu).$$

Since $\psi(p/q)$ is real, the imaginary part of the right member vanishes identically, so that

$$\psi\left(\frac{p}{q}\right)+\psi\left(\frac{q-p}{q}\right) = -2C-2\log q+\sum_{\mu=1}^{q-1}\cos\frac{2\pi\mu p}{q}\log\left(2-2\cos\frac{2\pi\mu}{q}\right),$$

which, together with

$$\psi\left(\frac{p}{q}\right)-\psi\left(\frac{q-p}{q}\right) = -\pi\cot\frac{\pi p}{q},$$

gives

$$\psi\left(\frac{p}{q}\right) = -C-\frac{\pi}{2}\cot\frac{\pi p}{q}-\log q+\frac{1}{2}\sum_{\mu=1}^{q-1}\cos\frac{2\pi\mu p}{q}\log\left(2-2\cos\frac{2\pi\mu}{q}\right).$$

Since

$$\cos\frac{2\pi(q-\mu)}{q}=\cos\frac{2\pi\mu}{q}\quad\text{and}\quad\cos\frac{2\pi(q-\mu)p}{q}=\cos\frac{2\pi\mu p}{q},$$

the summation on the right side may be extended from $\mu = 1$ to $\mu = \frac{q-1}{2}$ or $\frac{q}{2}$ according as q is odd or even. The final result thus becomes

$$\psi\left(\frac{p}{q}\right) = -C-\frac{\pi}{2}\cot\frac{\pi p}{q}-\log q+\sum_{\mu=1}^{q'}{}'\cos\frac{2\pi\mu p}{q}\log\left(2-2\cos\frac{2\pi\mu}{q}\right), \tag{32}$$

where $q' = q/2$ or $(q-1)/2$ according as q is even or odd, and the accent to the right of the summation sign indicates that the term corresponding to $\mu = q/2$ in the first case should be divided by 2. The proof given here is clearer and much simpler than that of Gauss,[12] to whom is due the remarkable theorem expressed by (32). As special cases, we find

$\frac{p}{q}$	$\psi\left(\frac{p}{q}\right)+C$
$\frac{1}{2}$	$-2\log 2$
$\frac{1}{3}$	$-\frac{\pi}{2\sqrt{3}} \quad -\frac{3}{2}\log 3$
$\frac{2}{3}$	$+\frac{\pi}{2\sqrt{3}} \quad -\frac{3}{2}\log 3$
$\frac{1}{4}$	$-\frac{\pi}{2} \quad -3\log 2$
$\frac{3}{4}$	$+\frac{\pi}{2} \quad -3\log 2$
$\frac{1}{5}$	$-\frac{\pi}{2}\sqrt{1+\frac{2}{\sqrt{5}}}-\frac{5}{4}\log 5-\frac{\sqrt{5}}{4}\log\frac{3+\sqrt{5}}{2}$
$\frac{2}{5}$	$-\frac{\pi}{2}\sqrt{1-\frac{2}{\sqrt{5}}}-\frac{5}{4}\log 5+\frac{\sqrt{5}}{4}\log\frac{3+\sqrt{5}}{2}$
$\frac{3}{5}$	$+\frac{\pi}{2}\sqrt{1-\frac{2}{\sqrt{5}}}-\frac{5}{4}\log 5+\frac{\sqrt{5}}{4}\log\frac{3+2\sqrt{5}}{2}$
$\frac{4}{5}$	$+\frac{\pi}{2}\sqrt{1+\frac{2}{\sqrt{5}}}-\frac{5}{4}\log 5-\frac{\sqrt{5}}{4}\log\frac{3+2\sqrt{5}}{2}$

10. The functions $P(s)$ and $Q(s)$. In § 2 we saw that $\Gamma(s)$ is a single-valued analytic function of s, having poles of the first order at the points $0, -1, -2, \cdots$, but holomorphic for all other finite values of s. In other words, $\Gamma(s+t)$ may be expanded in a power series in t for $|t|$ sufficiently small, unless s is zero or a negative integer. In the latter case, however, we have for sufficiently small values of $|t|$

$$\Gamma(-\nu+t)=\frac{(-1)^\nu}{\nu!}t^{-1}+a_0+a_1t+\cdots,$$

since

$$\lim_{t\to 0} t\Gamma(-\nu+t)=\lim_{t\to 0}\frac{\Gamma(1+t)}{(t-\nu)(t-\nu+1)\cdots(t-1)}=\frac{(-1)^\nu}{\nu!}.$$

We are thus led to form the infinite series

$$P(s)=\sum_0^\infty\frac{(-1)^\nu}{\nu!(\nu+s)},\tag{33}$$

which is uniformly convergent for all finite s and becomes infinite at the same points and in the same manner as $\Gamma(s)$. For when N is chosen so large that $N>|s|$, we have, for all $\nu>N$,

$$\left|\frac{(-1)^\nu}{\nu!(\nu+s)}\right| < \frac{1}{\nu!(\nu-N)},$$

which is independent of s and forms the general term of a convergent series. Hence it follows that $P(s)$ is a single-valued analytic function of s in the entire plane, which can become infinite at the points $0, -1, -2, \cdots$ only, but is holomorphic for all other finite s. For sufficiently small values of $|t|$, we have

$$P(-\nu+t) = \frac{(-1)^\nu}{\nu!}t^{-1} + \alpha_0 + \alpha_1 t + \cdots.$$

It is evident, therefore, that $\Gamma(s+t) - P(s+t)$ may be expanded in a power series in t for any finite s and for $|t|$ sufficiently small, and by a well-known theorem, $\Gamma(s) - P(s)$ is consequently expansible in a power series in s which converges for all finite s and is denoted by $Q(s)$. The result of the above investigation which is due to Prym[32] may be expressed by writing

$$(34) \qquad \Gamma(s) = P(s) + Q(s),$$

where $P(s)$ is defined by (33) while $Q(s)$ is an *entire transcendental*[33] *function* of s. We have

$$P(s+1) - sP(s) = \sum_0^\infty \frac{(-1)^\nu(\nu+1)}{(\nu+1)!(\nu+s+1)} - 1 - \sum_0^\infty \frac{(-1)^{\nu+1}s}{(\nu+1)!(\nu+s+1)}$$

$$= -1 + \sum_0^\infty \frac{(-1)^\nu}{(\nu+1)!} = -e^{-1},$$

and $P(s)$ consequently satisfies the functional equation

$$(35) \qquad P(s+1) = sP(s) - e^{-1},$$

while $Q(s)$, on account of (34) and (4a), will satisfy the analogous equation

$$(36) \qquad Q(s+1) = sQ(s) + e^{-1}.$$

When the distance of s from the nearest point of the negative real axis increases indefinitely, $P(s)$ approaches zero since, denoting this distance by ρ, we have

$$|P(s)| < \sum_0^\infty \frac{1}{\nu!}\cdot\frac{1}{\rho} = \frac{e}{\rho}.$$

Hence, a fortiori,

$$(37) \qquad \lim_{n\to\infty} \frac{P(n+s)}{(n-1)!n^s} = 0,$$

---

[32] Prym, F., Zur Theorie der Gammafunktion. Journal für Math., vol. 82 (1876), pp. 165–172.

[33] [That $Q(s)$ is transcendental follows most easily from equation (36) below, for supposing $Q(s)$ to be a polynomial of degree m, the polynomial $sQ(s)$, of degree $m+1$, would equal the polynomial $Q(s+1) - e^{-1}$, of degree m, which is clearly impossible.]

and by (34) and (4b)

$$\lim_{n\to\infty}\frac{Q(n+s)}{(n-1)!n^s}=1. \tag{38}$$

That equations (35) and (37) are sufficient to define $P(s)$ (Prym[32]), and consequently (36) and (38) sufficient to define $Q(s)$, is seen in the following way, which will also furnish a new expression for $P(s)$. From (35) we find upon division by $\Gamma(s+1)$

$$\frac{P(s)}{\Gamma(s)}-\frac{P(s+1)}{\Gamma(s+1)}=\frac{e^{-1}}{\Gamma(s+1)},$$

whence, replacing s by $s+1, s+2, \cdots, s+n-1$ and adding the resulting equations,

$$\frac{P(s)}{\Gamma(s)}-\frac{P(s+n)}{\Gamma(s+n)}=e^{-1}\sum_1^n\frac{1}{\Gamma(s+\nu)} \tag{39}$$

or

$$\begin{aligned}P(s+n)&=s(s+1)\cdots(s+n-1)\\&\quad\times\left[P(s)-e^{-1}\sum_1^n\frac{1}{s(s+1)\cdots(s+\nu-1)}\right],\end{aligned} \tag{39'}$$

and since

$$\lim_{n\to\infty}\frac{P(n+s)}{\Gamma(n+s)}=\lim_{n\to\infty}\frac{P(n+s)}{(n-1)!n^s}\times\lim_{n\to\infty}\frac{(n-1)!n^s}{\Gamma(n+s)}=0\times 1=0,$$

(39) gives for $n\to\infty$

$$\frac{P(s)}{\Gamma(s)}=e^{-1}\sum_1^\infty\frac{1}{\Gamma(s+\nu)} \tag{40}$$

or

$$P(s)=e^{-1}\sum_1^\infty\frac{1}{s(s+1)\cdots(s+\nu-1)} \tag{40'}$$

which is due to [Legendre and was rediscovered by] Bourguet.[34] Making $P(s)=\Gamma(s)-Q(s)$ in (39), we find

$$\frac{Q(s+n)}{\Gamma(s+n)}-\frac{Q(s)}{\Gamma(s)}=e^{-1}\sum_1^n\frac{1}{\Gamma(s+\nu)} \tag{41}$$

or

$$\begin{aligned}Q(s+n)&=s(s+1)\cdots(s+n-1)\\&\quad\times\left[Q(s)+e^{-1}\sum_1^n\frac{1}{s(s+1)\cdots(s+\nu-1)}\right].\end{aligned} \tag{41'}$$

---

[34] [Legendre, A. M., Exercices de calcul intégral, vol. 1 (Paris, 1811), pp. 339–343.] Bourguet, L., Développement en séries des intégrales Eulériennes. Annales de l'École Normale, ser. 2, vol. 10 (1881), pp. 175–233.

For $s \to 0$ it follows from (35) and (36) that $P(1) = 1 - e^{-1}$, $Q(1) = e^{-1}$, and for $s = 1$, $P(2) = P(1) - e^{-1} = 1 - 2e^{-1}$, $Q(2) = 2e^{-1}$, etc. It is easy to obtain general expressions for $P(n)$ and $Q(n)$ from (39′) and (41′); we find for $s \to 0$

$$\begin{aligned} P(n) &= (n-1)!\left[1 - e^{-1}\sum_{0}^{n-1}\frac{1}{\nu!}\right], \\ Q(n) &= (n-1)!e^{-1}\sum_{0}^{n-1}\frac{1}{\nu!}. \end{aligned} \tag{42}$$

From (36) it is seen that

$$Q(0) = \lim_{s\to 0}\frac{Q(1+s) - e^{-1}}{s} = \lim_{s\to 0}\frac{Q(1+s) - Q(1)}{s} = Q'(1),$$

which we shall learn to calculate in § 13. Making $s = -n$ in (41′) it is further seen that

$$\begin{aligned} Q(-n) &= \frac{(-1)^n}{n!}Q(0) + e^{-1} \\ &\times\left[\frac{1}{n} - \frac{1}{n(n-1)} + \frac{1}{n(n-1)(n-2)} - \cdots + \frac{(-1)^{n-1}}{n!}\right]. \end{aligned} \tag{43}$$

11. Solution of a functional equation. The form of equations (35) and (36) furnishes an occasion to consider the more general functional equation

$$f(s+1) = sf(s) - R(s), \tag{44}$$

where $R(s)$ is a polynomial in s. Let $S(s)$ be some definite solution, then $S(s+1) = sS(s) - R(s)$, and consequently

$$f(s+1) - S(s+1) = s[f(s) - S(s)],$$

or according to § 1, $f(s) - S(s) = p(s)\Gamma(s)$, where $p(s)$ is an arbitrary periodic function with the additive period unity. The *most general function* satisfying equation (44) is therefore

$$f(s) = S(s) + p(s)\Gamma(s).$$

To find a particular solution $S(s)$ we divide (44) by $\Gamma(s+1)$, whence

$$\frac{f(s)}{\Gamma(s)} - \frac{f(s+1)}{\Gamma(s+1)} = \frac{R(s)}{\Gamma(s+1)},$$

and consequently

$$\frac{f(s)}{\Gamma(s)} - \frac{f(s+n)}{\Gamma(s+n)} = \sum_{1}^{n}\frac{R(s+\nu-1)}{\Gamma(s+\nu)}.$$

In this equation, we now let n increase indefinitely, the resulting infinite series converging rapidly as shown by (4*b*), and we obtain

$$\frac{f(s)}{\Gamma(s)} - \lim_{n\to\infty} \frac{f(s+n)}{\Gamma(s+n)} = \sum_{1}^{\infty} \frac{R(s+\nu-1)}{\Gamma(s+\nu)},$$

where the limit on the left must exist and be finite. For the particular solution $f(s) = S(s)$ we assume this limit to be zero,[35] or

$$(45) \qquad \lim_{n\to\infty} \frac{S(s+n)}{\Gamma(s+n)} = \lim_{n\to\infty} \frac{S(s+n)}{(n-1)!\,n^s} = 0.$$

Hence we find

$$(46) \qquad \frac{S(s)}{\Gamma(s)} = \sum_{1}^{\infty} \frac{R(s+\nu-1)}{\Gamma(s+\nu)}.$$

Any polynomial of degree m may be written in the form

$$R(s) = a_0 + a_1 s + a_2 s(s-1) + \cdots + a_m s(s-1)\cdots(s-m+1),$$

as is readily seen by the method of undetermined coefficients, and since we have

$$\frac{R(s)}{\Gamma(s+1)} = \frac{a_0}{\Gamma(s+1)} + \frac{a_1}{\Gamma(s)} + \frac{a_2}{\Gamma(s-1)} + \cdots + \frac{a_m}{\Gamma(s-m+1)},$$

it follows from (40) that

$$\sum_{\nu=1}^{\infty} \frac{R(s+\nu-1)}{\Gamma(s+\nu)} = e \sum_{\mu=0}^{m} a_\mu \frac{P(s-\mu)}{\Gamma(s-\mu)}.$$

But in consequence of (39), $P(s-\mu)/\Gamma(s-\mu)$ may be written

$$\frac{P(s-\mu)}{\Gamma(s-\mu)} = \frac{P(s)}{\Gamma(s)} + e^{-1} \sum_{\nu=1}^{\mu} \frac{1}{\Gamma(s-\mu+\nu)},$$

and it is therefore seen at once that

$$\sum_{\nu=1}^{\infty} \frac{R(s+\nu-1)}{\Gamma(s+\nu)} = \frac{eP(s)}{\Gamma(s)} \sum_{\mu=0}^{m} a_\mu + \sum_{\mu=1}^{m} a_\mu \sum_{\nu=1}^{\mu} \frac{1}{\Gamma(s-\mu+\nu)},$$

or by (46), upon multiplication by $\Gamma(s)$,

$$(47) \quad S(s) = eP(s) \sum_{\mu=0}^{m} a_\mu + \sum_{\mu=1}^{m} a_\mu + \sum_{\mu=2}^{m} a_\mu \sum_{\nu=1}^{\mu-1} (s-1)(s-2)\cdots(s-\mu+\nu),$$

whereby the proposed functional equation is completely solved. Lindhagen[36] has derived this solution in a somewhat more complicated manner and without giving it in explicit form.

---

[35] [Carrying out a suggestion by the author, the translation differs here slightly from the original, in which the limit (45) is assumed to be a constant K, not necessarily zero.]

[36] Lindhagen, A., Studier öfver Gammafunktionen. Thesis, Upsala, 1887.

It would lead us too far here to discuss the solution of the more general functional equation

$$f(s+1) = r(s)f(s) + R(s),$$

where $r(s)$ and $R(s)$ are rational functions, and regarding which the researches of Mellin[37] may be consulted.

12. Investigation of $P(s)$ for real values of s. It appears immediately on inspection of the expansion (40′) that $P(x)$ is *positive and decreasing for real, positive and increasing* values of x. When x falls within the interval $(0 \cdots 1)$, we therefore have $P(x+1) > P(2) = 1 - 2e^{-1}$, and by (35)

$$eP(x) = \frac{eP(x+1)+1}{x} > \frac{e-1}{x}, \qquad (0 < x < 1),$$

an inequality which will be used below. In general it follows from (39′) that

$$(\alpha) \qquad eP(x-1) = \frac{eP(x)+1}{x-1},$$

$$(\beta) \qquad eP(x-2) = \frac{eP(x)+x}{(x-1)(x-2)},$$

$$(\gamma) \qquad eP(x-4) = \frac{eP(x)+x+(x-1)(x-2)^2}{(x-1)(x-2)(x-3)(x-4)}.$$

For an x in the interval $(0 \cdots 1)$, the numerator on the right in (α) is positive, the denominator negative, and hence $P(x)$ is *negative in the interval* $(-1 \cdots 0)$; on the other hand, both numerator and denominator are positive in (β), and consequently $P(x)$ is *positive in the interval* $(-2 \cdots -1)$. To show that the same is true for (γ), we use the inequality given above, and obtain, the denominator being positive,

$$eP(x-4) > \frac{e-1+x^2+x(x-1)(x-2)^2}{x(x-1)(x-2)(x-3)(x-4)}.$$

Making $x = 1 - y$, $0 < y < 1$, the numerator of the last fraction equals

$$\begin{aligned} e - 1 + (1-y)^2 - y(1-y)(1+y)^2 &\geqq e - 1 + (1-y)^2 - \tfrac{1}{4}(1+y)^2 \\ &= e - \tfrac{7}{4}y - \tfrac{3}{4}y(1-y) \\ &\geqq e - \tfrac{7}{4} - \tfrac{3}{16} = e - \tfrac{31}{16} > 0, \end{aligned}$$

---

[37] Mellin, H., Zur Theorie der Gammafunktion. Acta Math., vol. 8 (1886), pp. 37–80. Über einen Zusammenhang zwischen gewissen linearen Differential- und Differenzengleichungen, ibid., vol. 9 (1887), pp. 137–166. [Zur Theorie der linearen Differenzengleichungen erster Ordnung, ibid., vol. 15 (1891), pp. 317–384. Über den Zusammenhang zwischen den linearen Differential- und Differenzengleichungen, ibid., vol. 25 (1902), pp. 139–164.]

since $y(1 - y) = \frac{1}{4} - (\frac{1}{2} - y)^2 \leqq \frac{1}{4}$. Hence $P(x)$ is likewise positive in the interval $(-4 \cdots -3)$. From (β) it is seen immediately that when $P(x)$ is *negative for a negative value of* x, *so is also* $P(x - 2)$, and consequently $P(x)$ is *negative in the intervals* $(-3 \cdots -2)$, $(-5 \cdots -4)$, $\cdots (-2n - 1 \cdots -2n) \cdots$.

It only remains to investigate the intervals $(-6 \cdots -5)$, $(-8 \cdots -7)$, $\cdots$. When $\epsilon > 0$ but as small as we please, (33) gives

$$P(-2n + \epsilon) = \frac{1}{(2n)!\epsilon} + \cdots; \qquad P(-2n + 1 - \epsilon) = \frac{1}{(2n-1)!\epsilon} + \cdots.$$

$P(x)$ must therefore be *positive in the latter intervals, provided* x *is sufficiently near either end of the interval.* But we may show that $P(x)$ will also *assume negative values in these intervals.* We have by (γ) and (β)

$$eP(-\tfrac{11}{2}) = \frac{eP(-\frac{3}{2}) - \frac{257}{8}}{\frac{5}{2}\cdot\frac{7}{2}\cdot\frac{9}{2}\cdot\frac{11}{2}} = \frac{eP(\frac{1}{2}) - \frac{755}{32}}{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}\cdot\frac{9}{2}\cdot\frac{11}{2}},$$

and since, by (40′),

$$eP(\tfrac{1}{2}) = 2\left(1 + \frac{1}{\frac{3}{2}} + \frac{1}{\frac{3}{2}\cdot\frac{5}{2}} + \cdots\right) < 2e,$$

$P(-\frac{11}{2})$ is visibly negative, and hence also $P(-\frac{15}{2})$, $\cdots P(-2n + \frac{1}{2})$, $\cdots$. That the negative values which $P(x)$ thus assumes in the intervals $(-6 \cdots -5)$, $(-8 \cdots -7)$, $\cdots$, cannot be large, appears from the equation

$$P(x) - \frac{1}{ex} = \frac{P(x+1)}{x},$$

which shows that when $P(x)$, $P(x + 1)$ and x are negative, we must have $P(x) - 1/ex > 0$ or $|P(x)| < 1/e|x|$.

Since $P(x)$ is a continuous function of x in all the intervals in question, it is clear that the equation $P(x) = 0$ must have at least one root in each of the intervals $(-\frac{11}{2} \cdots -5)$, $(-6 \cdots -\frac{11}{2})$, $\cdots$ $(-2n + \frac{1}{2} \cdots -2n + 1)$, $(-2n \cdots -2n + \frac{1}{2})$, $\cdots$ but none in the remaining intervals (Bourguet[38]).

Note. It is a defect of the preceding investigation that it gives no information regarding the possibility of more than one root in the interval considered.[39] Bourguet[40] has attempted to show that the equation

---

[38] Bourguet, L., Sur la fonction Eulérienne, Comptes rendus, vol. 96 (1883), pp. 1307–1310, and Acta Math., vol. 2 (1883), pp. 296–298.

[39] [That each of the intervals in question contains exactly one root has been shown recently by C. N. Haskins, On the zeros of the function $P(x)$, complementary to the incomplete Gamma function, Trans. Am. Math. Soc., vol. 16 (1915), pp. 405–412. For a much simpler proof, which also shows that there are exactly four imaginary roots, see T. H. Gronwall, Sur les zéros des fonctions $P(z)$ et $Q(z)$ associées à la fonction gamma, Ann. de l'Ecole Normale, 1916.]

[40] Bourguet, L., Sur la théorie des intégrales Eulériennes, Comptes rendus, vol. 96 (1883), pp. 1487–1490.

$P(x) = 0$ cannot have more than 4 imaginary roots, but his proof is not correct. We may note in this connection that it is still undecided whether the equation $Q(s) = 0$ has any roots or not.[41, 42] That there are none when the real part of $s - 1$ is negative, was shown by Lindhagen.[36] We cannot enter further upon these questions which require more powerful methods than those at our disposal here, but on another occasion, the author proposes to examine the equation $P(s) = k$ for k real and show that for $k = 0$, the equation has exactly 4 imaginary roots.

13. Calculation of the coefficients in some power series. By (19) we have

$$\psi(1 + s) = -C + \sum_0^\infty \left(\frac{1}{\nu + 1} - \frac{1}{\nu + 1 + s}\right),$$

whence, expanding each term in powers of s for $|s| < 1$,

$$\psi(1 + s) = -S_1 + S_2 s - S_3 s^2 + S_4 s^3 - \cdots, \tag{48}$$

$$S_1 = C, \qquad S_n = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \cdots, \qquad (n > 1). \tag{48'}$$

The coefficients S_2 to S_{35} were computed by Legendre[43] to 16 decimals (15 correct) and S_2 to S_{70} by Stieltjes[44] to 32 decimals (30 correct). To determine the coefficients in the expansion

$$\Gamma(1 + s) = 1 + a_1 s + a_2 s^2 + \cdots, \tag{49}$$

which will be convergent for $|s| < 1$, we recollect that

$$\Gamma'(1 + s) = \psi(1 + s)\Gamma(1 + s),$$

or

$$a_1 + 2a_2 s + 3a_3 s^2 + \cdots = (-S_1 + S_2 s - S_3 s^2 + \cdots)(1 + a_1 s + a_2 s^2 + \cdots)$$

whence, comparing the coefficients of s^n on both sides,

$$(n + 1)a_{n+1} = -S_1 a_n + S_2 a_{n-1} - \cdots + (-1)^{n+1} S_{n+1}, \tag{49'}$$

---

[41] Later note. In a paper read before the Mathematical Society of Copenhagen in 1893, the author has given approximate expressions for the roots of $Q(s) = 0$. This investigation has not yet been published [but will appear in an early number of these *Annals*].

[42] [The existence of an infinity of zeros of $Q(s)$ may be proved in the following manner, which is not elementary, but has the advantage of brevity. Assuming $Q(s) = 0$ to have only a finite number of roots, $s_1, s_2, \cdots, s_n$ (or none), the only roots of $Q(s) = e^{-1}$ will be $1, s_1 + 1, s_2 + 1, \cdots, s_n + 1$ since, by (36′), $Q(s) = e^{-1}$ is equivalent to $(s - 1)Q(s - 1) = 0$. But by a theorem due to Picard, an entire function which takes each of *two* different values (here 0 and e^{-1}) only a *finite* number of times, is a polynomial. Since $Q(s)$ cannot be a polynomial,[33] it follows that the equation $Q(s) = 0$ has an infinity of roots.]

[43] Legendre, A. M., Traité des fonctions elliptiques et des intégrales Eulériennes, vol. 2 (Paris, 1826), p. 432.

[44] Stieltjes, T. J., Table des valeurs des sommes S_k, Acta Math., vol. 10 (1887), pp. 299-302.

which may be used to compute the coefficients a_n numerically. For $n = 1, 2, \cdots$ we find successively

$$a_1 = -S_1, \qquad a_2 = \tfrac{1}{2}(S_1^2 + S_2), \qquad a_3 = -\tfrac{1}{6}(S_1^3 + 3S_1S_2 + 2S_3), \cdots,$$

which however are less suited to numerical computation than (49′). Writing $(-1)^n a_n'$ instead of a_n in (49′), the alternating signs disappear, and it is seen that if $a_1', a_2', \cdots, a_n'$ are positive, this must also be true of a'_{n+1}. But $a_1' = S_1 > 0$, hence all a_n' are positive, and a_n *has the sign of* $(-1)^n$. It is also easy to calculate the coefficients of

$$P(1 + s) = b_0 + b_1 s + b_2 s^2 + \cdots, \qquad (|s| < 1), \tag{50}$$

by expanding each term of

$$P(1 + s) = \sum_0^\infty \frac{(-1)^\nu}{\nu!(\nu + 1 + s)}.$$

We obtain

$$b_0 = 1 - e^{-1}, \qquad b_n = \sum_{\nu=0}^\infty \frac{(-1)^{\nu+n}}{\nu!(\nu + 1)^{n+1}}, \tag{50′}$$

and these expressions are quite convenient for numerical computation. The infinite series for b_n consists of numerically decreasing terms with alternating signs, and consequently b_n has the same sign as the first term of the series, or the sign of $(-1)^n$.

Since $Q(1 + s) = \Gamma(1 + s) - P(1 + s)$, the coefficients in the power series

$$Q(1 + s) = c_0 + c_1 s + c_2 s^2 + \cdots, \tag{51}$$

which converges for all finite s, are given by

$$c_0 = e^{-1}, \qquad c_n = a_n - b_n. \tag{51′}$$

Since $c_1 = Q'(1)$, this also determines $Q(0)$, as we saw at the end of § 10. The calculation of the coefficients in the expansion

$$\frac{1}{\Gamma(1 + s)} = 1 + d_1 s + d_2 s^2 + \cdots, \tag{52}$$

which will converge for all finite s, may be effected by comparison of the coefficients of s^n in

$$(1 + a_1 s + a_2 s^2 + \cdots)(1 + d_1 s + d_2 s^2 + \cdots) = 1,$$

whence

$$d_n + d_{n-1}a_1 + d_{n-2}a_2 + \cdots + a_n = 0, \tag{52′}$$

which gives, using the previous expressions for $a_1, a_2, a_3, \cdots$,

$$d_1 = S_1, \qquad d_2 = \tfrac{1}{2}(S_1^2 - S_2), \qquad d_3 = \tfrac{1}{6}(S_1^3 - 3S_1S_2 + 2S_3), \cdots.$$

Having once calculated the coefficients $a_1, a_2, a_3, \cdots$, (52′) serves to determine $d_1, d_2, d_3, \cdots$. When an independent determination is preferred, we may proceed as follows. By Taylor's theorem and (52) we find

$$\frac{1}{\Gamma(1+s+t)} = \frac{1}{\Gamma(1+s) + t\Gamma'(1+s) + \cdots}$$
$$= 1 + d_1(s+t) + d_2(s+t)^2 + \cdots$$

and comparing the coefficients of t on both sides of the last equality sign,

$$-\frac{1}{\Gamma(1+s)} \cdot \frac{\Gamma'(1+s)}{\Gamma(1+s)} = -\frac{1}{\Gamma(1+s)} \cdot \psi(1+s) = d_1 + 2d_2 s + 3d_3 s^2 + \cdots,$$

or

$$(1 + d_1 s + d_2 s^2 + \cdots)(S_1 - S_2 s + S_3 s^2 - \cdots) = d_1 + 2d_2 s + 3d_3 s^2 + \cdots,$$

whence finally

$$(52'') \qquad (n+1)d_{n+1} = S_1 d_n - S_2 d_{n-1} + \cdots + (-1)^n S_{n+1},$$

which is of the same form as (49′) except that the signs of $S_1, S_2, \cdots$ are changed. The coefficients of the power series for $\Gamma(1+s)$ were first computed by Jeffery.[45] The coefficients in the expansions of $1/\Gamma(2+s) = 1/(1+s)\Gamma(1+s)$, $1/\Gamma(s)$, $e^{Cs}/\Gamma(2+s)$, $\Gamma(2+s) = (1+s)\Gamma(1+s)$, $Q(s)$ and $eP(s)/\Gamma(s)$ were computed by Bourguet[34, 46] to 16 decimals, of which the last is doubtful.

In the papers quoted, the general recurrent formulas are derived in a somewhat more complicated manner than above.

14. Expansions in series of factorials. In the following, we shall expand various of the functions introduced previously in series of the form

$$\Sigma \frac{c_\nu}{s(s+1)\cdots(s+\nu-1)}.$$

To simplify, we introduce the notation

$$(s)_n = s(s+1)\cdots(s+n-1)$$

when n is a positive integer, while for $n = 0$ we write $(s)_0 = 1$. Furthermore we agree not to let s equal a negative integer or zero. It is then seen that

$$\frac{(s')_\nu}{(s)_\nu} - \frac{(s')_{\nu+1}}{(s)_{\nu+1}} = \frac{(s')_\nu}{(s)_\nu}\left(1 - \frac{s'+\nu}{s+\nu}\right) = \frac{(s')_\nu}{(s)_{\nu+1}}(s - s'),$$

---

[45] Jeffery, H. M., On the derivatives of the Gamma function, Quarterly Journal of Math., vol. 6 (1864), pp. 82–108.

[46] Bourguet, L., Sur les intégrales Eulériennes et quelques autres fonctions uniformes, Acta Math., vol. 2 (1883), pp. 261–295.

whence for $|s - s'| > 0$, making $\nu = 0, 1, \cdots, n-1$ and adding,

$$(\alpha)\qquad \frac{1}{s-s'}\left[1-\frac{(s')_n}{(s)_n}\right]=\sum_0^{n-1}\frac{(s')_\nu}{(s)_{\nu+1}}.$$

Since $(s)_n = \Gamma(n+s)/\Gamma(s)$, and consequently

$$\frac{(s')_n}{(s)_n}=\frac{\Gamma(s)}{\Gamma(s')}\cdot\frac{\Gamma(s'+n)}{(n-1)!n^{s'}}\cdot\frac{(n-1)!n^{s}}{\Gamma(s+n)}\cdot n^{s'-s},$$

it is seen at once that, denoting the real part of s by $\Re(s)$,

$$\lim_{n\to\infty}\frac{(s')_n}{(s)_n}=\begin{cases}0\\ \text{indeterminate, as } \Re(s-s')\\ \infty\end{cases}\begin{cases}>0\\ =0.\\ <0\cdot\end{cases}$$

On account of (α) we therefore have the expansion, convergent for $\Re(s-s') > 0$,

$$(\beta)\qquad \frac{1}{s-s'}=\sum_0^{\infty}\frac{(s')_\nu}{(s)_{\nu+1}}=\frac{1}{s}+\frac{s'}{s(s+1)}+\frac{s'(s'+1)}{s(s+1)(s+2)}+\cdots$$

which is due to Stirling.[21] For $\Re(s-s') = 0$ the series will oscillate (between finite limits) and diverge for $\Re(s-s') < 0$. We have excluded above the case $s' = s$, where it is seen at once that the series reduces to $\Sigma\, 1/(s+\nu)$ and consequently diverges.

It does not appear from the preceding that the series is uniformly convergent, but this will be shown presently to be true for all values of s and s' subject to the conditions: s' finite, $\Re(s-s') \geqq \epsilon$, $|s|, |s+1|, |s+2|, \cdots \geqq \rho$, where ϵ and ρ are positive, but as small as we please.

We suppose these conditions fulfilled and write $s = x + yi$, $s' = x' + y'i$. The positive quantity A may then be chosen so large that $A > |s'|$, and consequently $x' > -A$, $|y'| < A$. Now taking an integer N satisfying the inequality

$$N > \frac{1}{\epsilon}A^2 + A,$$

we have for $\nu \geqq N$

$$\nu + x' \geqq N + x' > A + x' > 0$$

and

$$|\nu + s'| - (\nu + x') < \frac{\epsilon}{2},$$

since

$$(\nu+x')\left(\sqrt{1+\left(\frac{y'}{\nu+x'}\right)^2}-1\right)<\frac{y'^2}{2(\nu+x')}<\frac{A^2}{2\left(\frac{1}{\epsilon}A^2+2(A+x')\right)}\cdot$$

Therefore

$$|\nu + s| - |\nu + s'| > \frac{\epsilon}{2},$$

since $|\nu + s| \geqq \nu + x \geqq \nu + x' + \epsilon$. We now write

$$u_n = \frac{(s')_n}{(s)_{n+1}}, \qquad \left|\frac{u_n}{u_{n+1}}\right| = \left|\frac{s + n + 1}{s' + n}\right|,$$

and the preceding inequality becomes, making $\nu = n + 1$,

$$|s' + n| \cdot \left|\frac{u_n}{u_{n+1}}\right| - |s' + n + 1| > \frac{\epsilon}{2},$$

or

$$|u_{n+1}| < \frac{2}{\epsilon}(|s' + n| \cdot |u_n| - |s' + n + 1| \cdot |u_{n+1}|),$$

whence

$$|u_{n+1}| + |u_{n+2}| + \cdots < \frac{2}{\epsilon}|s' + n| \cdot |u_n| = \frac{2}{\epsilon}\prod_0^n \left|\frac{s' + \nu}{s + \nu}\right|.$$

But we have for all values of ν, and in particular for $\nu < N$,

$$\left|\frac{s' + \nu}{s + \nu}\right| < \frac{A + \nu}{\rho},$$

and for $\nu \geqq N$,

$$\left|\frac{s' + \nu}{s + \nu}\right| < \frac{|s' + \nu|}{|s' + \nu| + \frac{\epsilon}{2}} < \frac{\nu + A}{\nu + A + \frac{\epsilon}{2}}.$$

Consequently (if $n > N$),

$$\frac{2}{\epsilon}\prod_0^n \left|\frac{s' + \nu}{s + \nu}\right| < \frac{2}{\epsilon}\prod_0^{N-1}\left(\frac{A + \nu}{\rho}\right)\prod_N^n\left(\frac{\nu + A}{\nu + A + \frac{\epsilon}{2}}\right)$$

$$< \frac{2A(A + 1) \cdots (A + N - 1)}{\epsilon\rho^N \prod_N^n \left(1 + \frac{\epsilon}{2(\nu + A)}\right)},$$

which is *independent* of s and s' and converges to zero for n increasing. It is thus shown that Σu_n or the right side of (β) is ABSOLUTELY *and* UNIFORMLY *convergent when* s' *is finite,* $\Re(s - s') \geqq \epsilon > 0$ *and* $|s|$, $|s + 1|$, $|s + 2|$, $\cdots \geqq \rho > 0$.

On the basis of this result it would be easy to show that the boundaries of the regions of conditional as well as absolute convergence of a series of the form $\Sigma\, c_\nu/(s)_\nu$ are straight lines perpendicular to the real axis, and it is also possible to state the necessary and sufficient conditions for the expansibility of a function in such a series. As we shall have no occasion to use these results in the present paper, I shall not go beyond this indication.[47]

Assuming $| s | > | s' |$ and $\Re(s - s') > 0$, (β) may also be written

$$(\beta') \qquad \sum_0^\infty \frac{s'^\nu}{s^{\nu+1}} = \sum_0^\infty \frac{(s')_\nu}{(s)_{\nu+1}}.$$

The right hand series being uniformly convergent, it is permissible to expand in powers of s' and compare with the left side. Writing

$$(s')_n = C_0^n s'^n + C_1^n s'^{n-1} + \cdots + C_{n-1}^n s', \qquad C_0^n = 1,$$

where the first coefficients C_ν^n are readily calculated,[48] we find

$$(\gamma) \qquad \frac{1}{s^{\nu+1}} = \sum_{\mu=0}^\infty \frac{C_\mu^{\mu+\nu}}{(s)_{\mu+\nu+1}},$$

which will converge absolutely for $\Re(s) > 0$.*

---

[47] [An excellent account of this general theory is given by E. Landau, Über die Grundlagen der Theorie der Fakultätenreihen, Sitzungsber. Akad. München, vol. 36 (1906), pp. 151–218.]

[48] [Using, for instance, the identity $(s')_n = (s' + n - 1)(s')_{n-1}$, which gives the recurrent formula

$$C_\nu^n = C_\nu^{n-1} + (n - 1)C_{\nu-1}^{n-1}.]$$

* {We will give the following direct proof that (γ) converges for $\Re(s) > 0$. The ratio of consecutive terms in the series is

$$\frac{C_\mu^{\nu+\mu}}{C_{\mu-1}^{\nu+\mu-1}(\mu + s + \nu)} = \frac{\mu + (\nu - 1) + C_\mu^{\nu+\mu-1}/C_{\mu-1}^{\nu+\mu-1}}{\mu + (s + \nu)}.$$

But, as we shall prove, $\lim\limits_{\mu\to\infty} C_\mu^{\nu+\mu-1}/C_{\mu-1}^{\nu+\mu-1} = 0$. Hence the series will converge (and that absolutely) if $\Re(s) + \nu - (\nu - 1) > 1$, that is, if $\Re(s) > 0$.

That $\lim\limits_{\mu\to\infty} C_\mu^{\nu+\mu-1}/C_{\mu-1}^{\nu+\mu-1} = 0$ may be shown as follows: $C_\mu^{\nu+\mu-1}$ and $C_{\mu-1}^{\nu+\mu-1}$ are the sums of all the products of the numbers $1, 2, \cdots, \nu + \mu - 2$, taken μ at a time, and $\mu - 1$ at a time, respectively. Hence $C_\mu^{\nu+\mu-1}$ may be written

$$C_\mu^{\nu+\mu-1} = \Sigma n_1 n_2 \cdots n_\mu, \qquad 1 \leqq n_1 < n_2 < \cdots < n_\mu \leqq \nu + \mu - 2.$$

To each term $n_1 n_2 \cdots n_\mu$ in this sum there corresponds a set of μ products in $C_{\mu-1}^{\nu+\mu-1}$ whose sum is

$$n_1 n_2 \cdots n_\mu \left(\frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_\mu}\right).$$

But if we form this sum for each term in $\Sigma n_1 n_2 \cdots n^\mu$ in turn, each product in $C_{\mu-1}^{\nu+\mu-1}$ will be taken $\nu - 1$ times, since between the number of products in $C_\mu^{\nu+\mu-1}$ and that in $C_{\mu-1}^{\nu+\mu-1}$ the relation $\mu\binom{\nu+\mu-2}{\mu} = (\nu - 1)\binom{\nu+\mu-2}{\mu-1}$ exists, where $\binom{n}{r}$ denotes the number of combinations of n letters taken r at a time. We therefore have

$$\frac{C_\mu^{\nu+\mu-1}}{(\nu - 1)C_{\mu-1}^{\nu+\mu-1}} = \frac{\Sigma n_1 n_2 \cdots n_\mu}{\Sigma n_1 n_2 \cdots n_\mu \left(\frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_\mu}\right)} < \frac{1}{\frac{1}{\nu - 1} + \frac{1}{\nu} + \cdots + \frac{1}{\nu + \mu - 2}}.$$

But the limit, as μ approaches ∞, of the last member of this inequality is 0. The same must therefore be true of $C_\mu^{\nu+\mu-1}/C_{\mu-1}^{\nu+\mu-1}$.}

When we have a series

$$f(s) = \sum_0^\infty \frac{a_\nu}{s^{\nu+1}}$$

converging absolutely for $| s | > r$, this may be transformed in a series of factorials with the aid of (γ). We then obtain a double series, absolutely convergent for $\Re(s) > r$, which becomes, by combining the coefficients (Stirling[21]),

$$(\delta) \qquad f(s) = \frac{a_0}{(s)_1} + \frac{a_1}{(s)_2} + \frac{a_2 + a_1}{(s)_3} + \frac{a_3 + 3a_2 + 2a_1}{(s)_4} + \cdots,$$

where the general term is

$$(\epsilon) \qquad \frac{A_\nu}{(s)_{\nu+1}} = \frac{a_\nu + C_1^\nu a_{\nu-1} + C_2^\nu a_{\nu-2} + \cdots + C_{\nu-1}^\nu a_1}{(s)_{\nu+1}}$$

or symbolically

$$\frac{a(a+1)\cdots(a+\nu-1)}{(s)_{\nu+1}},$$

which could also be derived directly from (β') by symbolic calculation. A direct proof that (δ) is *absolutely and uniformly* convergent for

$$\Re(s) \geqq r + \epsilon,$$

where ϵ is positive but arbitrarily small, is reached by observing that, since $\Sigma\, a_\nu / s^{\nu+1}$ converges for $| s | = r + \epsilon/2$, there exists a finite and positive M such that $| a_\nu | < M(r + \epsilon/2)^\nu$ for $\nu = 0, 1, 2, \cdots$. Hence

$$\left| \frac{A_\nu}{(s)_{\nu+1}} \right| < M \cdot \frac{\left(r + \frac{\epsilon}{2}\right)_\nu}{| (s)_{\nu+1} |},$$

and the theorem follows by comparison with the series (β) in which we make $s' = r + \epsilon/2$.

The identity

$$\frac{1}{(s+\nu)_m} - \frac{1}{(s+\nu+1)_m} = \frac{m}{(s+\nu)_{m+1}}$$

gives

$$(\zeta) \qquad \sum_{\nu=0}^{n-1} \frac{1}{(s+\nu)_{m+1}} = \frac{1}{m}\left[\frac{1}{(s)_m} - \frac{1}{(s+n)_m}\right],$$

whence we find for $m \geqq 1$ and $n \to \infty$ (Stirling[21])

$$(\eta) \qquad \sum_{\nu=0}^\infty \frac{1}{(s+\nu)_{m+1}} = \frac{1}{m}\frac{1}{(s)_m}.$$

It is readily seen that this series is absolutely and uniformly convergent for $\Re(s) \geqq -A$ and $| s |, | s+1 |, | s+2 |, \cdots \geqq \rho$, where A is finite and $\rho > 0$ but as small as we please.

Applying the transformation (ζ) to the series (δ), it is seen that, making $a_0 = 0$,

$$f(s) + f(s+1) + \cdots + f(s+n-1) = \sum_{\nu=1}^{\infty} \frac{A_\nu}{\nu}\left[\frac{1}{(s)_\nu} - \frac{1}{(s+n)_\nu}\right].$$

We have, however,

$$\frac{A_\nu}{\nu(s+n)_\nu} = \left(1 + \frac{s}{\nu}\right)\frac{s}{s+n}\,\frac{(s+1)_{\nu-1}}{(s+n+1)_{\nu-1}}\cdot\frac{A_\nu}{(s)_{\nu+1}};$$

here $\Re(s) > r$, s is finite, $|s+\mu|/|s+n+\mu| < 1$ for $\mu = 1, 2, \cdots$ so that

$$\frac{(s+1)_{\nu-1}}{(s+n+1)_{\nu-1}} < 1,$$

and consequently

$$\left|\frac{A_\nu}{\nu(s+n)_\nu}\right| < \left|1 + \frac{s}{\nu}\right| \cdot \frac{|s|}{|s+n|} \cdot \left|\frac{A_\nu}{(s)_{\nu+1}}\right|.$$

Since $\Sigma\,|A_\nu/(s)_{\nu+1}|$ is convergent, $|1 + s/\nu|$ finite, and $|s|/|s+n| \to 0$ as $n \to \infty$, we therefore find

$$\lim_{n\to\infty} \sum_{\nu=1}^{\infty} \frac{A_\nu}{\nu(s+n)_\nu} = 0,$$

whence finally

$$(\vartheta) \qquad f(s) + f(s+1) + f(s+2) + \cdots = \sum_{\nu=1}^{\infty} \frac{A_\nu}{\nu(s)_\nu}$$

for $\Re(s) > r$, which is also due to Stirling.[21, 49]

These formulas may be applied immediately to the calculation of $\omega(s)$. It will be remembered from §§ 4 and 5 that

$$\omega(s) = \sum_{0}^{\infty}\left[(s + \nu + \tfrac{1}{2})\log\left(1 + \frac{1}{s+\nu}\right) - 1\right],$$

and that for $|s| > 1$

$$(s + \tfrac{1}{2})\log\left(1 + \frac{1}{s}\right) - 1 = \sum_{1}^{\infty} \frac{(-1)_\nu^{\nu+1}}{2(\nu+1)(\nu+2)s^{\nu+1}};$$

consequently, by (δ) and (ϵ),

$$(s + \tfrac{1}{2})\log\left(1 + \frac{1}{s}\right) - 1 = \sum_{1}^{\infty} \frac{k_\nu}{(s)_{\nu+1}},$$

$$k_\nu = \frac{(-1)^{\nu+1}\nu}{2(\nu+1)(\nu+2)} + \frac{(-1)^\nu(\nu-1)}{2\nu(\nu+1)}\,C_1^\nu + \cdots + \tfrac{1}{12}C_{\nu-1}^\nu,$$

---

[49] [The series to the right in (ϑ) is absolutely and uniformly convergent for $\Re(s) \geqq r + \epsilon$. For as shown above, $|A_\nu| < M(r + \epsilon/2)_\nu$, whence

$$\left|\frac{A_\nu}{\nu(s)_\nu}\right| < M\,\frac{r + \frac{\epsilon}{2} + \nu - 1}{\nu}\,\frac{\left(r + \frac{\epsilon}{2}\right)_{\nu-1}}{(s)_\nu} < M\left(r + \frac{\epsilon}{2} + 1\right)\frac{\left(r + \frac{\epsilon}{2}\right)_{\nu-1}}{(s)_\nu},$$

and our statement is proved by comparison with the series (β) in which we make $s' = r + \epsilon/2$.

and from (ϑ) we finally obtain

$$\omega(s) = \sum_1^\infty \frac{k_\nu}{\nu(s)_\nu} \tag{53}$$

which converges absolutely for $\Re(s) > 1$[50] and is due to Binet.[24] It is evident that $| k_\nu | < \frac{1}{12}(1)_\nu = \nu!/12$.

---

[50] [The above analysis does not give the complete domain of convergence. Actually (53) converges absolutely and uniformly for $\Re(s) \geqq \epsilon > 0$ (and consequently equals the holomorphic function $\omega(s)$ not only for $\Re(s) > 1$ but for $\Re(s) > 0$), but is divergent when $\Re(s) < 0$. For observing that $\mu/2(\mu + 1)(\mu + 2) = 1/(\mu + 2) - 1/2(\mu + 1)$, we have

$$k_\nu = - \sum_{\mu=1}^{\nu} \left(\frac{(-1)^{\mu+2}}{\mu + 2} + \frac{(-1)^{\mu+1}}{2(\mu+1)} \right) C_{\nu-\mu}^\nu = - \int_0^{-1} \sum_{\mu=1}^{\nu} (s^{\mu+1} + \tfrac{1}{2}s^\mu) C_{\nu-\mu}^\nu \, ds,$$

or by reference to the identity defining the coefficients $C_{\nu-\mu}^\nu$,

$$- k_\nu = \int_0^{-1} (s + \tfrac{1}{2})(s)_\nu ds.$$

Replacing s by $- s$ in the integral, we obtain, for $\nu > 2$,

$$- k_\nu = \int_0^1 (\tfrac{1}{2} - s)s(1 - s)(2 - s)\cdots(\nu - 1 - s)ds.$$

But for $0 < s < 1$, we have $| 1/2 - s | < 1, s < 1, 1 - s < 1, 2 - s < 2, \cdots, \nu - 1 - s < \nu - 1$' so that

$$| k_\nu | < \int_c^1 1 \cdot 2 \cdot \; \cdots (\nu - 1)ds = (\nu - 1)!,$$

and for $\Re(s) \geqq \epsilon$ and $\nu > 2$,

$$\left| \frac{k_\nu}{\nu(s)_\nu} \right| < \frac{(\nu - 1)(1)_{\nu-2}}{\nu \, | s | \, | (s + 1)_{\nu-1} |} < \frac{1}{\epsilon} \cdot \frac{(1)_{\nu-2}}{| (s + 1)_{\nu-1} |};$$

in consequence of what was proved above for the series (β), it follows that (53) is absolutely and uniformly convergent for $\Re(s + 1) \geqq 1 + \epsilon$ or $\Re(s) \geqq \epsilon$. Now transform the expression for $- k_\nu$ by decomposing the integral in two with the limits $0, \frac{1}{2}$ and $\frac{1}{2}, 1$ respectively, and replacing s by $1 - s$ in the latter. Thus

$$- k_\nu = \int_0^{1/2} (\tfrac{1}{2} - s)s(1 - s)[(2 - s) \cdots (\nu - 1 - s) - (1 + s) \cdots (\nu - 2 + s)]ds,$$

where the integrand is evidently positive, and therefore

$$| k_\nu | > \int_0^\delta (\tfrac{1}{2} - s)s(1 - s)[(2 - s) \cdots (\nu - 1 - s) - (1 + s) \cdots (\nu - 2 + s)]ds$$

for a positive δ less than $\frac{1}{2}$ and as small as we please. But for $0 < s < \delta < \frac{1}{3}$ we have $1 + s < \frac{4}{5}(2 - s)$, $2 + s < 3 - s$, $\cdots$, $\nu - 2 + s < \nu - 1 - s$, so that $(1 + s) \cdots (\nu - 2 + s) < \frac{4}{5}(2 - s) \cdots (\nu - 1 - s)$, and consequently

$$\begin{aligned} \left| \frac{k_\nu}{\nu} \right| &> \frac{1}{5} \int_0^\delta (\tfrac{1}{2} - s)s(1 - s)(2 - s) \cdots (\nu - 2 - s) \frac{\nu - 1 - s}{\nu} ds \\ &> \frac{1}{5} \int_0^\delta (\tfrac{1}{2} - \tfrac{1}{3})s(1 - \delta)(2 - \delta) \cdots (\nu - 2 - \delta) \cdot \tfrac{1}{2}ds \\ &= \frac{1}{120} \delta^2(1 - \delta)(2 - \delta) \cdots (\nu - 2 - \delta) = \frac{\delta}{72} |(-\delta)_{\nu-1}|. \end{aligned}$$

Hence $\left| \frac{k_\nu}{\nu(s)_\nu} \right| > \frac{\delta}{120} \left| \frac{(-\delta)_{\nu-1}}{(s)_\nu} \right| \rightarrow \infty$ as $\nu \rightarrow \infty$ when $\Re(s) < - \delta$, so that (53) diverges when $\Re(s) < - \delta$].

The first coefficients are $k_1 = \frac{1}{12}, k_2 = 0, k_3 = -\frac{1}{120}, k_4 = -\frac{1}{30}, \cdots$. When s is large, the computation of only a few terms in (53) gives a very close approximation.

In the same way we obtain from

$$(s + \tfrac{1}{2}) \log\left(1 + \frac{1}{s}\right) - 1 = \sum_1^\infty \frac{\nu}{2(\nu + 1)(\nu + 2)(s + 1)^{\nu+1}},$$

which is valid for $| s + 1 | > 1$,

$$(s + \tfrac{1}{2}) \log\left(1 + \frac{1}{s}\right) - 1 = \sum_1^\infty \frac{k_\nu}{(s + 1)_{\nu+1}},$$

$$k_\nu = \frac{\nu}{2(\nu + 1)(\nu + 2)} + \frac{\nu - 1}{2\nu(\nu + 1)} C_1^\nu + \cdots + \tfrac{1}{12} C_{\nu-1}^\nu,$$

and

$$\omega(s) = \sum_1^\infty \frac{k_\nu}{\nu(s + 1)_\nu}, \tag{54}$$

which is also due to Binet[24] and converges absolutely and uniformly when $\Re(s) \geqq \epsilon > 0$.[51] The first coefficients are $k_1 = \frac{1}{12}, k_2 = \frac{1}{6}, k_3 = \frac{59}{120}, k_4 = \frac{29}{15}$ $\cdots$. Making $s = n$, a positive integer, in (53) and (54), we obtain [see (14)] the convergent series

$$\log(n!) - (n + \tfrac{1}{2}) \log n + n - \log \sqrt{2\pi}$$

$$= \frac{1}{12n} - \frac{1}{360n(n + 1)(n + 2)} - \frac{1}{120n(n + 1)(n + 2)(n + 3)} - \cdots$$

$$= \frac{1}{12(n + 1)} + \frac{1}{12(n + 1)(n + 2)} + \frac{59}{360(n + 1)(n + 2)(n + 3)}$$

$$+ \frac{29}{60(n + 1)(n + 2)(n + 3)(n + 4)} + \cdots.$$

Remembering that (§ 8)

$$\omega^*(s) = \sum_0^\infty \left[\frac{1}{s + \nu} - \log\left(1 + \frac{1}{s + \nu}\right)\right],$$

and that for $| s | > 1$, on account of (δ) and (ϵ),

$$\frac{1}{s} - \log\left(1 + \frac{1}{s}\right) = \sum_1^\infty \frac{(-1)^{\nu+1}}{(\nu + 1)s^{\nu+1}} = \sum_1^\infty \frac{k_\nu}{(s)_{\nu+1}},$$

where

$$k_\nu = \frac{(-1)^{\nu+1}}{\nu + 1} + \frac{(-1)^\nu}{\nu} C_1^\nu + \cdots + \tfrac{1}{2} C_{\nu-1}^\nu,$$

we find by (ϑ)

$$\omega^*(s) = \sum_1^\infty \frac{k_\nu}{\nu(s)_\nu}, \tag{55}$$

---

[51] [Using the method of [50], the series is readily seen to diverge for $\Re(s) < 0$.]

which will converge absolutely for $\Re(s) > 1$,[52] and in which the first coefficients are $k_1 = \frac{1}{2}$, $k_2 = \frac{1}{6}$, $k_3 = \frac{1}{4}$, $k_4 = \frac{19}{30}$, $\cdots$. Similarly, using the expansion

$$\frac{1}{s} - \log\left(1 + \frac{1}{s}\right) = \sum_{1}^{\infty} \frac{\nu}{(\nu+1)(s+1)^{\nu+1}} = \sum_{1}^{\infty} \frac{k_\nu}{(s+1)_{\nu+1}},$$

which converges for $\Re(s) > 0$, we find

$$k_\nu = \frac{\nu}{\nu+1} + \frac{\nu-1}{\nu} C_1^\nu + \cdots + \tfrac{1}{2} C_{\nu-1}^\nu$$

and

$$\omega^*(s) = \sum_{1}^{\infty} \frac{k_\nu}{\nu(s+1)_\nu}, \tag{56}$$

which is absolutely and uniformly convergent for $\Re(s) \geqq \epsilon > 0$.[53] The first coefficients are $k_1 = \frac{1}{2}$, $k_2 = \frac{7}{6}$, $k_3 = \frac{15}{4}$, $k_4 = \frac{469}{30}$, $\cdots$. As special cases of (55) and (56), we note the convergent series

$$\begin{aligned} 1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \frac{1}{n-1} - \log n - C \\ = -\frac{1}{2n} - \frac{1}{12n(n+1)} - \frac{1}{12n(n+1)(n+2)} \\ - \frac{19}{120n(n+1)(n+2)(n+3)} - \cdots \\ = -\frac{1}{2(n+1)} - \frac{7}{12(n+1)(n+2)} - \frac{5}{4(n+1)(n+2)(n+3)} \\ - \frac{469}{120(n+1)(n+2)(n+3)(n+4)} - \cdots. \end{aligned}$$

We shall finally give an application to the function $\psi(s)$. By (19‴) we have

$$\psi(s) - \psi(s - s') = \sum_{0}^{\infty} \left(\frac{1}{s - s' + \nu} - \frac{1}{s + \nu} \right),$$

and since for $\Re(s - s') > 0$

$$\frac{1}{s - s'} - \frac{1}{s} = \sum_{1}^{\infty} \frac{(s')_\nu}{(s)_{\nu+1}},$$

---

[52] [Actually the series converges absolutely and uniformly for $\Re(s) \geqq \epsilon > 0$, but diverges for $\Re(s) < 0$. For we have, by the method of [50],

$$k_\nu = \int_0^1 s(1-s)(2-s) \cdots (\nu - 1 - s)\,ds,$$

whence $0 < k_\nu < (\nu - 1)!$, and for $0 < \delta < 1$,

$$\frac{k_\nu}{\nu} > \int_0^\delta s(1-\delta)(2-\delta) \cdots (\nu - 2 - \delta) \frac{\nu - 1 - \delta}{\nu}\,ds > \frac{\delta}{4} \left| (-\delta)_{\nu-1} \right|.]$$

[53] [But divergent for $\Re(s) < 0$, since $k_\nu > \frac{1}{2} + \frac{1}{2}C_1^\nu + \cdots + \frac{1}{2}C_{\nu-1}^\nu = \frac{1}{2}(\nu - 1)!$]

it follows from (ϑ) that

$$\psi(s) - \psi(s - s') = \sum_{1}^{\infty} \frac{(s')_\nu}{\nu (s)_\nu},$$

or replacing s' by $-s'$,

$$(57) \quad \psi(s + s') - \psi(s) = \frac{s'}{s} - \frac{s'(s' - 1)}{2s(s + 1)} + \frac{s'(s' - 1)(s' - 2)}{3s(s + 1)(s + 2)} - \cdots,$$

which converges absolutely and uniformly for $\Re(s + s') \geqq \epsilon > 0$.[54] Assuming s' to be a positive integer, we obtain the finite series

$$\frac{1}{s} + \frac{1}{s + 1} + \cdots + \frac{1}{s + n - 1} = \frac{n}{s} - \frac{n(n - 1)}{2s(s + 1)} + \cdots$$

15. Numerical computation of the gamma function.[55] Let us now consider a practical method of computing $\Gamma(s)$ for real values of s. It is easily shown that $\Gamma(s)$ is known when its values in *any* interval of length $\frac{1}{2}$ are known.

From (5) we have

$$(58) \qquad L(s) = \log \Gamma(s) = -Cs - \log s + \sum_{n=1}^{\infty} \left\{ \frac{s}{n} - \log\left(1 + \frac{s}{n}\right) \right\}.$$

It is seen that it is permissible to differentiate the series on the right termwise; so that

$$L'(s) = -C - \frac{1}{s} + \sum_{1}^{\infty} \left\{ \frac{1}{n} - \frac{1}{s + n} \right\}.$$

This may be written

$$L'(s) = -C + \sum_{1}^{\infty} \left\{ \frac{1}{n} - \frac{1}{s + n - 1} \right\}.$$

These relations hold for $s \neq 0, -1, -2, \cdots$. In general we have

$$(59) \qquad L^{(r)}(s) = (-1)^r (r - 1)! \sum_{1}^{\infty} \frac{1}{(s + n - 1)^r}, \qquad r > 1.$$

In particular

$$(60) \qquad L'(1) = -C,$$

$$(61) \qquad L^{(r)}(1) = (-1)^r (r - 1)! \Sigma \frac{1}{n^r} = (-1)^r (r - 1)! S_r,$$

where S_r has the meaning of (48′).

Using Taylor's development about $s = 1$, we have

$$\log \Gamma(s) = L(s) = L(1) + \frac{s - 1}{1!} L'(1) + \frac{(s - 1)^2}{2!} L''(1) + \cdots;$$

---

[54] [But evidently diverges for $\Re(s + s') < 0$.]

[55] This section has been added in the translation.

or setting $s = 1 + x$ and taking account of (60) and (61)

$$\log \Gamma(1+x) = -Cx + \sum_{n=2}^{\infty} \frac{(-1)^n}{n} S_n x^n. \tag{62}$$

We show now that this relation is valid for $-\frac{1}{2} \leq x \leq \frac{1}{2}$, by proving that

$$R_m = \frac{x^m}{m!} L^{(m)}(1 + \theta x), \qquad 0 < \theta < 1$$

converges to 0, as $m \to \infty$.

For, if $-\frac{1}{2} \leq x \leq \frac{1}{2}$, then

$$|R_m| \leq \left\{ \left| \frac{x}{1+\theta x} \right|^m + \sum_2^{\infty} \frac{x^m}{|n + \theta x|^m} \right\} \cdot \frac{1}{m} < \left\{ 1 + \sum_2^{\infty} \frac{1}{2^m (n-1)^m} \right\} \frac{1}{m} \to 0.$$

As a matter of fact the region of convergence could be shown to be $-1 < x \leq 1$, but for our purposes it suffices to know that the series converges for $-\frac{1}{2} \leq x \leq \frac{1}{2}$. We have for any x in $(-\frac{1}{2}, \frac{1}{2})$

$$\log (1+x) = x - \sum_2^{\infty} (-1)^n \frac{x^n}{n}.$$

This on adding and subtracting from (62) gives

$$\log \Gamma(1+x) = -\log(1+x) + (1-C)x + \sum_2^{\infty} (-1)^n (S_n - 1) \frac{x^n}{n}.$$

Changing here x into $-x$ gives

$$\log \Gamma(1-x) = -\log(1-x) - (1-C)x + \sum_2^{\infty} (S_n - 1) \frac{x^n}{n}.$$

Subtracting this from the foregoing gives

$$\log \Gamma(1+x) - \log \Gamma(1-x) = -\log \frac{1+x}{1-x} + 2(1-C)x - \sum_1^{\infty} \frac{x^{2m+1}}{2m+1} (S_{2m+1} - 1).$$

From (6), substituting therein $\Gamma(1+s)/s$ for $\Gamma(s)$ and setting $s = x$,

$$\log \Gamma(1+x) + \log \Gamma(1-x) = \log \frac{\pi x}{\sin \pi x}.$$

This with the preceding relation gives

$$\log \Gamma(1+x) = (1-C)x + \tfrac{1}{2} \log \frac{\pi x}{\sin \pi x} - \tfrac{1}{2} \log \frac{1+x}{1-x} - \frac{1}{2} \sum_1^{\infty} (S_{2m+1} - 1) \frac{x^{2m+1}}{2m+1}. \tag{63}$$

This series is due to Legendre. It converges rapidly for $0 \leq x \leq \frac{1}{2}$ and enables one to compute $\Gamma(s)$ in the interval $1 \leq s \leq \frac{3}{2}$. The value of $\Gamma(s)$ for other real values of s may then be obtained as already observed.